6/23 第14回　　201810869　永井 るい

例1：具体的なクレブシュゴルダン係数

$$\boxed{\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0}$$

$j_1 = \frac{1}{2}$, $j_2 = \frac{1}{2}$, 右$=1$として、$m = m_1 + m_2$ごとの基底を次のようにとって

$$\Psi_1 = \left(\left| \tfrac{1}{2}, \tfrac{1}{2} \right\rangle \right) = \left(|\uparrow\uparrow\rangle \right)$$

$$\Psi_0 = \left(\left| \tfrac{1}{2}, -\tfrac{1}{2} \right\rangle, \left| -\tfrac{1}{2}, \tfrac{1}{2} \right\rangle \right) = \left(|\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle \right)$$

$$\Psi_{-1} = \left(\left| -\tfrac{1}{2}, -\tfrac{1}{2} \right\rangle \right) = \left(|\downarrow\downarrow\rangle \right)$$

$$j=1 \begin{cases} |11\rangle = |\uparrow\uparrow\rangle = \Psi_1 (1) \\ |10\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \right) = \Psi_0 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \\ |1-1\rangle = |\downarrow\downarrow\rangle = \Psi_{-1}(1) \end{cases}$$

$$j=0 \rightarrow |00\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right) = \Psi_0 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

これはクレブシュゴルダン係数を使えば次のようにかける

$$|1,1\rangle = \left| \tfrac{1}{2}, \tfrac{1}{2} \right\rangle \left\langle \tfrac{1}{2}, \tfrac{1}{2} \middle| 1,1 \right\rangle$$

$$|1,0\rangle = \left| \tfrac{1}{2}, -\tfrac{1}{2} \right\rangle \left\langle \tfrac{1}{2}, -\tfrac{1}{2} \middle| 1,0 \right\rangle + \left| -\tfrac{1}{2}, \tfrac{1}{2} \right\rangle \left\langle -\tfrac{1}{2}, \tfrac{1}{2} \middle| 1,0 \right\rangle$$

$$|0,0\rangle = \left| \tfrac{1}{2}, -\tfrac{1}{2} \right\rangle \left\langle \tfrac{1}{2}, -\tfrac{1}{2} \middle| 0,0 \right\rangle + \left| -\tfrac{1}{2}, \tfrac{1}{2} \right\rangle \left\langle -\tfrac{1}{2}, \tfrac{1}{2} \middle| 0,0 \right\rangle$$

$$|1,-1\rangle = \left| -\tfrac{1}{2}, -\tfrac{1}{2} \right\rangle \left\langle -\tfrac{1}{2}, -\tfrac{1}{2} \middle| -1,-1 \right\rangle$$

まとめ

表2.1：$\langle j_1 m_1 j_2 m_2 | jm \rangle$, $j_1 = 1$, $j_2 = \frac{1}{2}$, $j = \frac{1}{2}$

j	m	m_1	m_2	$\langle j_1 m_1 j_2 m_2 \mid jm \rangle$
1	1	$\frac{1}{2}$	$\frac{1}{2}$	1
1	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{\sqrt{2}}{2}$
1	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$
1	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	1

表2.2：$\langle j_1 m_1 j_2 m_2 | jm \rangle$, $j_1 = 0$, $j_2 = \frac{1}{2}$, $j = \frac{1}{2}$

j	m	m_1	m_2	$\langle j_1 m_1 j_2 m_2 \mid jm \rangle$
0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{\sqrt{2}}{2}$
0	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$

例2: 具体的なクレブシュゴルダン係数

$$\boxed{1\otimes\frac{1}{2}=\frac{3}{2}\oplus\frac{1}{2}}$$

m一定の基底:

$$\Psi_{\frac{3}{2}}=\left(|1\rangle|\tfrac{1}{2}\rangle\right)$$
$$\Psi_{\frac{1}{2}}=\left(|1\rangle|-\tfrac{1}{2}\rangle,\ |0\rangle|\tfrac{1}{2}\rangle\right)$$
$$\Psi_{-\frac{1}{2}}=\left(|-1\rangle|\tfrac{1}{2}\rangle,\ |0\rangle|-\tfrac{1}{2}\rangle\right)$$
$$\Psi_{-\frac{3}{2}}=\left(|-1\rangle|-\tfrac{1}{2}\rangle\right)$$

$m=1+\frac{1}{2}=\frac{3}{2}$ は $|1\rangle|\frac{1}{2}\rangle$ のみで

$$\left|\tfrac{3}{2}\,\tfrac{3}{2}\right\rangle=|1\rangle|\tfrac{1}{2}\rangle=\Psi_{\frac{3}{2}}(1)$$ ★

$J_-|\frac{3}{2}\,\frac{3}{2}\rangle=\sqrt{3}|\frac{3}{2}\,\frac{1}{2}\rangle$

$j=\frac{3}{2}$ の公式より

$$\left|\tfrac{3}{2}\,\tfrac{1}{2}\right\rangle=\frac{1}{\sqrt{3}}J_-\left|\tfrac{3}{2}\,\tfrac{3}{2}\right\rangle=\frac{1}{\sqrt{3}}(J_{1-}+J_{2-})|1\rangle|\tfrac{1}{2}\rangle$$
$$=\frac{1}{\sqrt{3}}\left[J_{1-}|1\rangle|\tfrac{1}{2}\rangle+|1\rangle J_{2-}|\tfrac{1}{2}\rangle\right]$$
$$=\frac{1}{\sqrt{3}}\left[\sqrt{2}|0\rangle|\tfrac{1}{2}\rangle+|1\rangle|-\tfrac{1}{2}\rangle\right]$$
$$=\Psi_{\frac{1}{2}}\begin{pmatrix}\frac{\sqrt{3}}{3}\\ \frac{\sqrt{6}}{3}\end{pmatrix}$$ ★

続いて

$J_-|\frac{3}{2}\,\frac{1}{2}\rangle=2|\frac{3}{2}\,-\frac{1}{2}\rangle$

$$\left|\tfrac{3}{2}\,-\tfrac{1}{2}\right\rangle=\frac{1}{2}J_-\left|\tfrac{3}{2}\,\tfrac{1}{2}\right\rangle=\frac{1}{2}(J_{1-}+J_{2-})\left[\frac{\sqrt{6}}{3}|0\rangle|\tfrac{1}{2}\rangle+\frac{\sqrt{3}}{3}|1\rangle|-\tfrac{1}{2}\rangle\right]$$
$$=\frac{\sqrt{6}}{6}\left[J_{1-}|0\rangle|\tfrac{1}{2}\rangle+|0\rangle J_{2-}|\tfrac{1}{2}\rangle\right]+\frac{\sqrt{3}}{6}\left[J_-|1\rangle|-\tfrac{1}{2}\rangle+|1\rangle J_{2-}|-\tfrac{1}{2}\rangle\right]$$
$$=\frac{\sqrt{6}}{6}\left[\sqrt{2}|-1\rangle|\tfrac{1}{2}\rangle+|0\rangle|-\tfrac{1}{2}\rangle\right]+\frac{\sqrt{3}}{6}\sqrt{2}|0\rangle|-\tfrac{1}{2}\rangle$$
$$=\frac{\sqrt{3}}{3}|-1\rangle|\tfrac{1}{2}\rangle+\frac{\sqrt{6}}{3}|0\rangle|-\tfrac{1}{2}\rangle$$
$$=\Psi_{-\frac{1}{2}}\begin{pmatrix}\frac{\sqrt{3}}{3}\\ \frac{\sqrt{6}}{3}\end{pmatrix}$$ ★

更に

$J_-|\frac{3}{2}\,-\frac{1}{2}\rangle=\sqrt{3}|\frac{3}{2}\,-\frac{3}{2}\rangle$

$$\left|\tfrac{3}{2}\,-\tfrac{3}{2}\right\rangle=\frac{1}{\sqrt{3}}J_-\left|\tfrac{3}{2}\,-\tfrac{1}{2}\right\rangle=\frac{1}{\sqrt{3}}(J_{1-}+J_{2-})\left[\frac{\sqrt{3}}{3}|-1\rangle|\tfrac{1}{2}\rangle+\frac{\sqrt{6}}{3}|0\rangle|-\tfrac{1}{2}\rangle\right]$$
$$=\frac{1}{3}|-1\rangle J_{2-}|\tfrac{1}{2}\rangle+\frac{\sqrt{2}}{3}J_{1-}|0\rangle|-\tfrac{1}{2}\rangle=\frac{1}{3}|-1\rangle|-\tfrac{1}{2}\rangle+\frac{\sqrt{2}}{3}\sqrt{2}|-1\rangle|-\tfrac{1}{2}\rangle$$
$$=\Psi_{-\frac{3}{2}}(1)$$ ★

以上の手続きをもう少し見通しよく行う。まず、最初に基底の変換をつぎのように計算しておく。

$$J_-\Psi_{\frac{3}{2}}=(J_{1-}+J_{2-})\left(|1\rangle|\tfrac{1}{2}\rangle\right)=\left(\sqrt{2}|0\rangle|\tfrac{1}{2}\rangle+|1\rangle|-\tfrac{1}{2}\rangle\right)=\Psi_{\frac{1}{2}}\begin{pmatrix}1\\ \sqrt{2}\end{pmatrix}$$
$$J_-\Psi_{\frac{1}{2}}=(J_{1-}+J_{2-})\left(|1\rangle|-\tfrac{1}{2}\rangle,\ |0\rangle|\tfrac{1}{2}\rangle\right)=\left(\sqrt{2}|0\rangle|-\tfrac{1}{2}\rangle,\ \sqrt{2}|-1\rangle|\tfrac{1}{2}\rangle+|0\rangle|-\tfrac{1}{2}\rangle\right)$$
$$=\Psi_{-\frac{1}{2}}\begin{pmatrix}0&\sqrt{2}\\ \sqrt{2}&1\end{pmatrix}$$
$$J_-\Psi_{-\frac{1}{2}}=(J_{1-}+J_{2-})\left(|-1\rangle|\tfrac{1}{2}\rangle,\ |0\rangle|-\tfrac{1}{2}\rangle\right)=\left(|-1\rangle|-\tfrac{1}{2}\rangle,\ \sqrt{2}|-1\rangle|-\tfrac{1}{2}\rangle\right)=\Psi_{-\frac{3}{2}}(1,\sqrt{2})$$
$$\Psi_{-\frac{3}{2}}=\left(|-1\rangle|-\tfrac{1}{2}\rangle\right)$$

これを用いれば

$$\left|\tfrac{3}{2}\,\tfrac{3}{2}\right\rangle = \Psi_{\frac{3}{2}}(1)$$

$$\left|\tfrac{3}{2}\,\tfrac{1}{2}\right\rangle = \frac{1}{\sqrt{3}} J_- \Psi_{\frac{3}{2}}(1) = \Psi_{\frac{1}{2}}\begin{pmatrix} \frac{\sqrt{3}}{6} \\ \frac{\sqrt{6}}{3} \end{pmatrix}$$

$$\left|\tfrac{3}{2}\,-\tfrac{1}{2}\right\rangle = \frac{1}{2} J_- \left|\tfrac{3}{2}\,\tfrac{1}{2}\right\rangle = \frac{1}{2} J_- \Psi_{\frac{1}{2}}\begin{pmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{3} \end{pmatrix} = \Psi_{-\frac{1}{2}}\begin{pmatrix} 0 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} \frac{\sqrt{3}}{6} \\ \frac{\sqrt{6}}{6} \end{pmatrix} = \Psi_{-\frac{1}{2}}\begin{pmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{3} \end{pmatrix}$$

$$\left|\tfrac{3}{2}\,-\tfrac{3}{2}\right\rangle = \frac{1}{\sqrt{3}} J_- \left|\tfrac{3}{2}\,-\tfrac{1}{2}\right\rangle = \frac{1}{\sqrt{3}} J_- \Psi_{-\frac{1}{2}}\begin{pmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{3} \end{pmatrix} = \Psi_{-\frac{3}{2}}(1, \sqrt{2})\begin{pmatrix} \frac{1}{3} \\ \frac{\sqrt{2}}{3} \end{pmatrix} = \Psi_{-\frac{3}{2}}(1)$$

表2.3：$\langle j_1 m_1 j_2 m_2 | j m \rangle$, $j_1 = 1$, $j_2 = \frac{1}{2}$, $j = \frac{3}{2}$

j	m	m_1	m_2	$\langle j_1 m_1 j_2 m_2 \| j m \rangle$
$\frac{3}{2}$	$\frac{3}{2}$	1	$\frac{1}{2}$	1
$\frac{3}{2}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{3}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{6}}{3}$
$\frac{3}{2}$	$-\frac{1}{2}$	-1	$\frac{1}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{3}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{\sqrt{6}}{3}$
$\frac{3}{2}$	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	1

$\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle \perp \left|\tfrac{3}{2}\,\tfrac{1}{2}\right\rangle$

次に $j = \frac{1}{2}$ の固有状態を構成する。$m = m_1 + m_2 = \frac{1}{2}$ の空間は $\Psi_{\frac{1}{2}}$ で2次元の基底で張られているが、$\Psi_{\frac{3}{2}}$ から J_- を作用させて、$\left|\tfrac{3}{2}\,\tfrac{1}{2}\right\rangle$ として、1状態既に使われているので、$\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle$ をそれと直交するように定める。

$$\left|\tfrac{3}{2}\,\tfrac{1}{2}\right\rangle = \Psi_{\frac{1}{2}}\begin{pmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{3} \end{pmatrix}$$

$$\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle = \Psi_{\frac{1}{2}}\begin{pmatrix} a \\ b \end{pmatrix}, \quad a^2 + b^2 = 1$$

$$\left\langle \tfrac{3}{2}\,\tfrac{1}{2} \middle| \tfrac{1}{2}\,\tfrac{1}{2} \right\rangle = \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{3}\right) \Psi_{\frac{1}{2}}^{\dagger} \Psi_{\frac{1}{2}}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\sqrt{3}}{3} a + \frac{\sqrt{6}}{3} b = 0$$

$a = -b\sqrt{2}$, $a^2 + b^2 = 3b^2 = 1$, $a > 0$ とすれば $a = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$, $b = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$

$$\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle = \Psi_{\frac{1}{2}}\begin{pmatrix} \frac{\sqrt{6}}{3} \\ -\frac{\sqrt{3}}{3} \end{pmatrix} \star$$

$$\left|\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle = J_- \left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle = \Psi_{-\frac{1}{2}}\begin{pmatrix} 0 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} \frac{\sqrt{6}}{3} \\ -\frac{\sqrt{3}}{3} \end{pmatrix} = \Psi_{-\frac{1}{2}}\begin{pmatrix} -\frac{\sqrt{6}}{3} \\ \frac{\sqrt{3}}{3} \end{pmatrix} \star$$

表2.4：$\langle j_1 m_1 j_2 m_2 | j m \rangle$, $j_1 = 1$, $j_2 = \frac{1}{2}$, $j = \frac{1}{2}$

j	m	m_1	m_2	$\langle j_1 m_1 j_2 m_2 \| j m \rangle$
$\frac{1}{2}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	$\frac{\sqrt{6}}{3}$
$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{\sqrt{3}}{3}$
$\frac{1}{2}$	$-\frac{1}{2}$	-1	$\frac{1}{2}$	$-\frac{\sqrt{6}}{3}$
$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{\sqrt{3}}{3}$

射影演算子の方法　もう少し見通しよく行う別法

2次元の$\Phi_{\frac{1}{2}}$で張られる空間で規格化されている$|v\rangle = |\frac{3}{2}\frac{1}{2}\rangle = \Phi_{\frac{1}{2}} v$, $v = \begin{pmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{3} \end{pmatrix}$と直交する状態$|v_\perp\rangle$はv方向への射影 $\boxed{P = |v\rangle\langle v|}$ を用いて、任意の(一般の)状態$|t\rangle$から次式で構成できる。

$$|v_\perp\rangle = P'|t\rangle, \quad P' = 1 - P, \quad P = |v\rangle\langle v| \qquad P^2 = P, \ (P')^2 = P'$$

$$\because \langle v|v_\perp\rangle = \langle v|(1-P)|t\rangle = \langle v|t\rangle - \langle v|v\rangle\langle v|t\rangle = 0$$

ここで$|t\rangle = \Phi_{\frac{1}{2}} t$, $t = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$とすれば、$\langle v|t\rangle = v^\dagger \Phi_{\frac{1}{2}}^\dagger \Phi_{\frac{1}{2}} t = \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{3}\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\sqrt{3}}{3}$

$$|v_\perp\rangle = |t\rangle - |v\rangle\langle v|t\rangle = \Phi_{\frac{1}{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \Phi_{\frac{1}{2}}\begin{pmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{3} \end{pmatrix}\frac{\sqrt{3}}{3} = \Phi_{\frac{1}{2}}\begin{pmatrix} \frac{2}{3} \\ -\frac{\sqrt{2}}{3} \end{pmatrix}$$

$\langle v_\perp|v_\perp\rangle = \frac{6}{9} = \frac{2}{3}$だから、規格化して

$$|\tfrac{1}{2}\tfrac{1}{2}\rangle = |v_\perp\rangle\frac{1}{\sqrt{\langle v_\perp|v_\perp\rangle}} = \Phi_{\frac{1}{2}}\begin{pmatrix} \frac{\sqrt{6}}{3} \\ -\frac{\sqrt{3}}{3} \end{pmatrix}$$

と以前の計算に一致する。

例3：具体的なクレブシュゴルダン係数

$$\boxed{1 \otimes 1 = 2 \oplus 1 \oplus 0}$$

m一定の基底：

$$\begin{aligned}
\Phi_2 &= (|1\rangle|1\rangle) \\
\Phi_1 &= (|1\rangle|0\rangle, |0\rangle|1\rangle) \\
\Phi_0 &= (|-1\rangle|0\rangle, |0\rangle|0\rangle, |-1\rangle|1\rangle) \\
\Phi_{-1} &= (|-1\rangle|0\rangle, |0\rangle|-1\rangle) \\
\Phi_{-2} &= (|-1\rangle|-1\rangle)
\end{aligned}$$

基底の変換

$$J_-\Phi_2 = (J_{1-} + J_{2-})(|1\rangle|1\rangle) = (\sqrt{2}|0\rangle|1\rangle + \sqrt{2}|1\rangle|0\rangle) = \Phi_1\begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix}$$

$$J_-\Phi_1 = (J_{1-} + J_{2-})(|1\rangle|0\rangle, |0\rangle|1\rangle) = (\sqrt{2}|0\rangle|0\rangle + \sqrt{2}|1\rangle|-1\rangle, \sqrt{2}|-1\rangle|1\rangle + \sqrt{2}|0\rangle|0\rangle)$$

$$= \Phi_0\begin{pmatrix} \sqrt{2} & 0 \\ \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{2} \end{pmatrix}$$

$$\begin{aligned}
J_-\Phi_0 &= (J_{1-} + J_{2-})(|1\rangle|-1\rangle, |0\rangle|0\rangle, |-1\rangle|1\rangle) \\
&= (\sqrt{2}|0\rangle|-1\rangle, \sqrt{2}|-1\rangle|0\rangle + \sqrt{2}|0\rangle|-1\rangle, \sqrt{2}|-1\rangle|0\rangle) \\
&= \Phi_{-1}\begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 0 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
J_-\Phi_{-1} &= (J_{1-} + J_{2-})(|-1\rangle|0\rangle, |0\rangle|-1\rangle) = (\sqrt{2}|-1\rangle|-1\rangle, \sqrt{2}|-1\rangle|-1\rangle) \\
&= \Phi_{-2}(\sqrt{2}, \sqrt{2})
\end{aligned}$$

$$|22\rangle = \Phi_2(1)$$

$$|21\rangle = \frac{1}{2}J_-\Phi_2(1) = \Phi_1\begin{pmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{pmatrix}$$

$$|20\rangle = \frac{1}{\sqrt{6}}J_-\Phi_1\begin{pmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{pmatrix} = \frac{1}{\sqrt{6}}\Phi_0\begin{pmatrix}\sqrt{2} & 0\\ \sqrt{2} & \sqrt{2}\\ 0 & \sqrt{2}\end{pmatrix}\begin{pmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{pmatrix} = \Phi_0\begin{pmatrix}\frac{\sqrt{6}}{6}\\ \frac{\sqrt{6}}{3}\\ \frac{\sqrt{6}}{6}\end{pmatrix}$$

$$|2\,{-1}\rangle = \frac{1}{\sqrt{6}}J_-\Phi_0\begin{pmatrix}\frac{\sqrt{6}}{6}\\ \frac{\sqrt{6}}{3}\\ \frac{\sqrt{6}}{6}\end{pmatrix} = \Phi_{-1}\begin{pmatrix}0 & \sqrt{2} & \sqrt{2}\\ \sqrt{2} & \sqrt{2} & 0\end{pmatrix}\begin{pmatrix}\frac{1}{6}\\ \frac{1}{3}\\ \frac{1}{6}\end{pmatrix} = \Phi_{-1}\begin{pmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{pmatrix}$$

$$|2\,{-2}\rangle = \frac{1}{2}J_-\Phi_{-1} = \frac{1}{2}\Phi_{-2}(\sqrt{2},\sqrt{2})\begin{pmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{pmatrix} = \Phi_{-2}(1).$$

表2.5：$\langle j_1m_1j_2m_2|jm\rangle$, $j_1=1$, $j_2=1$, $j=2$

j	m	m_1	m_2	$\langle j_1m_1j_2m_2\|jm\rangle$
2	2	1	1	1
2	1	0	0	$\sqrt{2}/2$
2	1	0	1	$\sqrt{2}/2$
2	0	1	-1	$\sqrt{6}/6$
2	0	0	0	$\sqrt{6}/3$
2	0	-1	1	$\sqrt{6}/6$
2	-1	-1	0	$\sqrt{2}/2$
2	-1	0	-1	$\sqrt{2}/2$
2	-2	-1	-1	1

次に、$j=1$の固有状態を構成する。Φ_1は2次元の基底であるが、$|v\rangle = |21\rangle = \Phi_1\begin{pmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{pmatrix}$として1つ使われているので、$|11\rangle$をそれと直交するように定める。

$P=|v\rangle\langle v|$として射影の方法を用いる。$|t\rangle = \Phi_1 t$, $t=\begin{pmatrix}1\\0\end{pmatrix}$とすれば、

$$|v_\perp\rangle = (1-P)|t\rangle = |t\rangle - |v\rangle\langle v|t\rangle = \Phi_1\begin{pmatrix}1\\0\end{pmatrix} - \Phi_1\begin{pmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{pmatrix}\frac{\sqrt{2}}{2} = \Phi_1\begin{pmatrix}\frac{1}{2}\\ -\frac{1}{2}\end{pmatrix}$$

規格化して

$$|11\rangle = |v_\perp\rangle\sqrt{2} = \Phi_1\begin{pmatrix}\frac{\sqrt{2}}{2}\\ -\frac{\sqrt{2}}{2}\end{pmatrix}$$

$$|10\rangle = \frac{1}{\sqrt{2}}J_-|11\rangle = \frac{1}{\sqrt{2}}J_-\Phi_1\begin{pmatrix}\frac{\sqrt{2}}{2}\\ -\frac{\sqrt{2}}{2}\end{pmatrix} = \frac{1}{\sqrt{2}}\Phi_0\begin{pmatrix}\sqrt{2} & 0\\ \sqrt{2} & \sqrt{2}\\ 0 & \sqrt{2}\end{pmatrix}\begin{pmatrix}\frac{\sqrt{2}}{2}\\ -\frac{\sqrt{2}}{2}\end{pmatrix} = \Phi_0\begin{pmatrix}\frac{\sqrt{2}}{2}\\ 0\\ -\frac{\sqrt{2}}{2}\end{pmatrix}$$

$$|1\,{-1}\rangle = \frac{1}{\sqrt{2}}J_-|10\rangle = \frac{1}{\sqrt{2}}J_-\Phi_0\begin{pmatrix}\frac{\sqrt{2}}{2}\\ 0\\ -\frac{\sqrt{2}}{2}\end{pmatrix} = \frac{1}{\sqrt{2}}\Phi_{-1}\begin{pmatrix}0 & \sqrt{2} & \sqrt{2}\\ \sqrt{2} & \sqrt{2} & 0\end{pmatrix}\begin{pmatrix}\frac{\sqrt{2}}{2}\\ 0\\ -\frac{\sqrt{2}}{2}\end{pmatrix} = \Phi_{-1}\begin{pmatrix}-\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{pmatrix}$$

表2.6：$\langle j_1 m_1 j_2 m_2 | jm \rangle$, $j_1 = 1$, $j_2 = 1$, $j = 1$

j	m	m_1	m_2	$\langle j_1 m_1 j_2 m_2 \mid jm \rangle$
1	1	1	0	$\sqrt{2}/2$
1	1	0	1	$-\sqrt{2}/2$
1	0	1	-1	$\sqrt{2}/2$
1	0	0	0	0
1	0	-1	1	$-\sqrt{2}/2$
1	-1	-1	0	$-\sqrt{2}/2$
1	-1	0	-1	$\sqrt{2}/2$

残りは Φ_0 に属する $|00\rangle$ である。$|00\rangle$ は $|u\rangle = |20\rangle = \Phi_0 \begin{pmatrix} \sqrt{6}/6 \\ \sqrt{6}/3 \\ \sqrt{6}/6 \end{pmatrix}$ と $|v\rangle = |10\rangle = \Phi_0 \begin{pmatrix} \sqrt{2}/2 \\ 0 \\ -\sqrt{2}/2 \end{pmatrix}$ とに垂直である。

任意の（例えば）$|t\rangle = \Phi_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ に対して

$$|u_\perp, v_\perp\rangle = (1-P)|t\rangle \propto |00\rangle$$

$$\because \langle u | u_\perp, v_\perp \rangle = \langle u|t\rangle - \langle u|(P_u + P_v)|t\rangle$$

$$= \langle u|t\rangle - \langle u|P_u|t\rangle = 0$$

$$\langle v | u_\perp, v_\perp \rangle = 0$$

$$\langle u|t\rangle = \left(\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}\right) \Phi_0^\dagger \Phi_0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \left(\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}\right) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{\sqrt{6}}{6}$$

$$P_u|t\rangle = |u\rangle\langle u|t\rangle = |u\rangle \frac{\sqrt{6}}{6} = \Phi_0 \begin{pmatrix} \frac{1}{6} \\ \frac{1}{3} \\ \frac{1}{6} \end{pmatrix}$$

$$\langle v|t\rangle = \left(\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}\right) \Phi_0^\dagger \Phi_0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \left(\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}\right) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{\sqrt{2}}{2}$$

$$P_v|t\rangle = |v\rangle\langle v|t\rangle = |v\rangle \frac{\sqrt{2}}{2} = \Phi_0 \begin{pmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix}$$

$$|u_\perp, v_\perp\rangle = |t\rangle - P_u|t\rangle - P_v|t\rangle = \Phi_0 \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{6} \\ \frac{1}{3} \\ \frac{1}{6} \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix} \right) = \Phi_0 \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

規格化して

$$|00\rangle = |u_\perp, v_\perp\rangle \sqrt{3} = \Phi_0 \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

表2.7：$\langle j_1 m_1 j_2 m_2 | jm \rangle$, $j_1 = 1$, $j_2 = 1$, $j = 0$

j	m	m_1	m_2	$\langle j_1 m_1 j_2 m_2 \mid jm \rangle$
0	0	1	-1	$1/\sqrt{3}$
0	0	0	0	$-1/\sqrt{3}$
0	0	-1	1	$1/\sqrt{3}$