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$[J_i, J_j] = i\hbar\varepsilon_{ijk}J_k$ ：一般の角運動量

→ $[J^2, J_z] = 0$ ：同時固有状態をとる

$$\begin{cases} J^2|jm\rangle = \hbar^2 j(j+1)|jm\rangle \\ J_z|jm\rangle = \hbar m|jm\rangle \end{cases}$$

規格直交性
$\langle jm|j'm'\rangle = \delta_{jj'}\delta_{mm'}$

完全性
$\sum_{jm}|jm\rangle\langle jm| = 1$

j, m の値には制限がある

→ 角運動量の量子化

$j = 0, 1, 2, 3 \cdots$
$\quad 1/2, 3/2, 5/2 \cdots$ の2パターン

それぞれのjについて

$m = -j, -j+1, \cdots, j-1, j$ の $2j+1$ 通り

$j=0$ なら $m=0$ の1通り

$j=1/2$ 〃 $m=-1/2, 1/2$ の2通り

$j=1$ 〃 $m=-1, 0, 1$ の3通り

昇降演算子

$J_+|jm\rangle = \hbar\sqrt{(j-m)(j+m+1)}\,|j, m+1\rangle$

$J_-|jm\rangle = \hbar\sqrt{(j+m)(j-m+1)}\,|j, m-1\rangle$

$J_\pm = J_x \pm iJ_y$

$J_+|jj\rangle = 0$

$J_-|j, -j\rangle = 0$

$m = \underbrace{-j \cdots j}_{2j+1個}$

特に $j=1/2$ の場合

$J_+|\frac{1}{2}\frac{1}{2}\rangle = 0$, $J_+|\frac{1}{2}, -\frac{1}{2}\rangle = \hbar|\frac{1}{2}, \frac{1}{2}\rangle$

$J_-|\frac{1}{2}\frac{1}{2}\rangle = \hbar|\frac{1}{2}, -\frac{1}{2}\rangle$, $J_-|\frac{1}{2}, -\frac{1}{2}\rangle = 0$

$j=0$ の場合

$J_+|1, m\rangle = \hbar\sqrt{(1-m)(2+m)}\,|1, m+1\rangle$, $J_-|1, m\rangle = \hbar\sqrt{(1+m)(2-m)}\,|1, m-1\rangle$

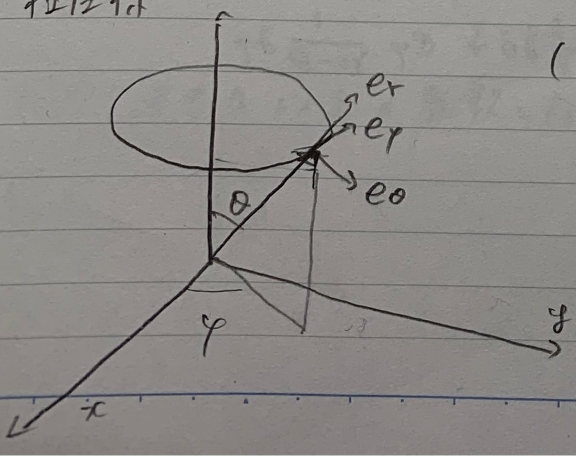
$J_+|1, 1\rangle = 0$, $J_+|1, 0\rangle = \hbar\sqrt{2}|1, 1\rangle$, $J_+|1, -1\rangle = \hbar\sqrt{2}|1, 0\rangle$

$J_-|1, 1\rangle = \hbar\sqrt{2}|1, 0\rangle$, $J_-|1, 0\rangle = \hbar\sqrt{2}|1, -1\rangle$, $J_-|1, -1\rangle = 0$

軌道角運動量と球面調和関数 （軌道角運動量 $\mathbf{L} = \mathbf{r}\times\mathbf{p}$）

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos\varphi\sin\theta \\ r\sin\varphi\sin\theta \\ r\cos\theta \end{pmatrix}, \quad \partial_r\mathbf{r} = \begin{pmatrix} \cos\varphi\sin\theta \\ \sin\varphi\sin\theta \\ \cos\theta \end{pmatrix}, \quad \partial_\theta\mathbf{r} = \begin{pmatrix} r\cos\varphi\cos\theta \\ r\sin\varphi\cos\theta \\ -r\sin\theta \end{pmatrix}, \quad \partial_\varphi\mathbf{r} = \begin{pmatrix} -r\sin\varphi\sin\theta \\ r\cos\varphi\sin\theta \\ 0 \end{pmatrix}$$

極座標



$$(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi) = \left(\frac{\partial_r\mathbf{r}}{h_r}, \frac{\partial_\theta\mathbf{r}}{h_\theta}, \frac{\partial_\varphi\mathbf{r}}{h_\varphi}\right)$$

$h_r = |\partial_r\mathbf{r}| = 1$

$h_\theta = |\partial_\theta\mathbf{r}| = r$

$h_\varphi = |\partial_\varphi\mathbf{r}| = r\sin\theta \geq 0$

$\mathbf{e}_i\cdot\mathbf{e}_j = \delta_{ij}$

$\mathbf{e}_r\times\mathbf{e}_\theta = \mathbf{e}_\varphi$

$\mathbf{e}_\theta\times\mathbf{e}_\varphi = \mathbf{e}_r$

$\mathbf{e}_\varphi\times\mathbf{e}_r = \mathbf{e}_\theta$

$(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi)$ ：規格直交化された右手系

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勾配演算子を極座標で表示する

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos\varphi\sin\theta \\ r\sin\varphi\sin\theta \\ r\cos\theta \end{pmatrix}$$

$$T = *D^{-1}, \quad * = TD, \quad \tilde{*} = \widetilde{TD} = D\tilde{T}$$

$$(\mathbf{e}_r\ \mathbf{e}_\theta\ \mathbf{e}_\varphi) = \left(\frac{\partial_r \mathbf{r}}{h_r},\ \frac{\partial_\theta \mathbf{r}}{h_r},\ \frac{\partial_\varphi \mathbf{r}}{h_r}\right) = \begin{pmatrix} \frac{1}{h_r}\frac{\partial x}{\partial r} & \frac{1}{h_\theta}\frac{\partial x}{\partial \theta} & \frac{1}{h_\varphi}\frac{\partial x}{\partial \varphi} \\ \frac{1}{h_r}\frac{\partial y}{\partial r} & \frac{1}{h_\theta}\frac{\partial y}{\partial \theta} & \frac{1}{h_\varphi}\frac{\partial y}{\partial \varphi} \\ \frac{1}{h_r}\frac{\partial z}{\partial r} & \frac{1}{h_\theta}\frac{\partial z}{\partial \theta} & \frac{1}{h_\varphi}\frac{\partial z}{\partial \varphi} \end{pmatrix} = T$$

$$\begin{array}{c} \\ = \\ \\ \end{array} \begin{array}{c} \begin{pmatrix} \cos\varphi\sin\theta & \cos\varphi\cos\theta & -\sin\varphi \\ \sin\varphi\sin\theta & \sin\varphi\cos\theta & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \\ \mathbf{e}_r \qquad\quad \mathbf{e}_\theta \qquad\quad \mathbf{e}_\varphi \end{array} \equiv T$$

$$* = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{pmatrix} D^{-1}$$

$$\tilde{T}T = E_3 \qquad \tilde{T} = T^{-1}, \quad \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$$

$$\begin{pmatrix} \tilde{\mathbf{e}}_r \\ \tilde{\mathbf{e}}_\theta \\ \tilde{\mathbf{e}}_\varphi \end{pmatrix} (\mathbf{e}_r\ \mathbf{e}_\theta\ \mathbf{e}_\varphi) = \begin{pmatrix} \tilde{\mathbf{e}}_r\mathbf{e}_r & \tilde{\mathbf{e}}_r\mathbf{e}_\theta & \tilde{\mathbf{e}}_r\mathbf{e}_\varphi \\ \tilde{\mathbf{e}}_\theta\mathbf{e}_r & \tilde{\mathbf{e}}_\theta\mathbf{e}_\theta & \tilde{\mathbf{e}}_\theta\mathbf{e}_\varphi \\ \tilde{\mathbf{e}}_\varphi\mathbf{e}_r & \tilde{\mathbf{e}}_\varphi\mathbf{e}_\theta & \tilde{\mathbf{e}}_\varphi\mathbf{e}_r \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E_3$$

$$D = \mathrm{diag}(h_r, h_\theta, h_\varphi) = \mathrm{diag}(1, r, r\sin\theta)$$

$$h_r = |\partial_r \mathbf{r}| = 1, \quad h_\theta = |\partial_\theta \mathbf{r}| = r, \quad h_\varphi = |\partial_\varphi \mathbf{r}| = r\sin\theta$$

一方 上記をみて

$$\begin{pmatrix} \partial_r \\ \partial_\theta \\ \partial_\varphi \end{pmatrix} = \overset{\tilde{*}}{\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} & \frac{\partial z}{\partial \varphi} \end{pmatrix}} \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} = D\tilde{T}\begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}$$

$$\rightarrow \partial_r = \frac{\partial x}{\partial r}\partial_x + \frac{\partial y}{\partial r}\partial_y + \frac{\partial z}{\partial r}\partial_z \quad など$$

逆に解いて $\tilde{T}^{-1} = T$

$$\nabla = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} = TD^{-1}\begin{pmatrix} \partial_r \\ \partial_\theta \\ \partial_\varphi \end{pmatrix} = T\begin{pmatrix} \partial_r \\ \frac{1}{r}\partial_\theta \\ \frac{1}{r\sin\theta}\partial_\varphi \end{pmatrix}$$

$$= \mathbf{e}_r\partial_r + \mathbf{e}_\theta\frac{1}{r}\partial_\theta + \mathbf{e}_\varphi\frac{1}{r\sin\theta}\partial_\varphi$$

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c.f. $Y_{lm} = \frac{1}{\sqrt{2\pi}} e^{im\varphi}\, \Theta_{lm}(\theta)$

$L_+ = \hbar e^{i\varphi}(\partial_\theta + i\cot\theta\,\partial_\varphi)$

$L_- = \hbar e^{-i\varphi}(-\partial_\theta + i\cot\theta\,\partial_\varphi)$

ここで θの関数 $f(\theta)$ に対して

$$L_+[e^{im\varphi}f(\theta)] = e^{im\varphi}\hbar e^{i\varphi}\left[\frac{df}{d\theta} - mf\cot\theta\right]$$

$$L_-[e^{im\varphi}f(\theta)] = -e^{im\varphi}\hbar e^{-i\varphi}\left[\frac{df}{d\theta} + mf\cot\theta\right]$$

これらを次の関係式に注意して書き直す

$$\frac{d}{d\theta} = \frac{d\cos\theta}{d\theta}\frac{d}{d\cos\theta} = -\sin\theta\frac{d}{d\cos\theta}, \quad \frac{d}{d\cos\theta} = -\sin^{-1}\theta\frac{d}{d\theta}$$

$$\frac{d\sin\theta}{d\cos\theta} = \frac{d(1-\cos^2\theta)^{1/2}}{d\cos\theta} = \frac{1}{2}\underbrace{(1-\cos^2\theta)^{-\frac{1}{2}}}_{=\sin\theta}(-2\cos\theta) = -\cot\theta$$

$$L_+[e^{im\varphi}f(\theta)] = -\hbar e^{i(m+1)\varphi}\sin^{m+1}\theta\frac{d}{d\cos\theta}[\sin^{-m}\theta f(\theta)]$$

$$-\sin^{-m-1}\theta\left(\frac{df}{d\theta} - mf\cot\theta\right) = -m\sin^{-m-1}\theta(-\cot\theta)f + \sin^{-m}\theta\left(-\sin^{-1}\theta\frac{df}{d\theta}\right) = -m\sin^{-m-1}\theta\frac{d\sin\theta}{d\cos\theta}f + \sin^{-m}\theta\frac{df}{d\cos\theta}$$

$$L_-[e^{im\varphi}f(\theta)] = \hbar e^{i(m-1)\varphi}\sin^{-(m-1)}\theta\frac{d}{d\cos\theta}[\sin^{m}\theta f(\theta)]$$

$$-\sin^{m-1}\theta\left(\frac{df}{d\theta} + mf\cot\theta\right) = m\sin^{m-1}\theta(-\cot\theta f) + \sin^{m}\theta\left(-\sin^{-1}\theta\frac{df}{d\theta}\right) = m\sin^{m-1}\theta\frac{d\sin\theta}{d\cos\theta}f + \sin^{m}\theta\frac{df}{d\cos\theta}$$

$$L_+[e^{im\varphi}f(\theta)] = -\hbar e^{i(m+1)\varphi}\underbrace{\sin^{m+1}\theta\frac{d}{d\cos\theta}[\sin^{-m}\theta f(\theta)]}_{\text{書ける}\ (m\to m+1)}$$

くりかえして

$$L_+^2[e^{im\varphi}f(\theta)] = -\hbar L_+\left[e^{i(m+1)\varphi}\sin^{m+1}\theta\frac{d}{d\cos\theta}[\sin^{-m}\theta f(\theta)]\right]$$

$$= (-\hbar)^2 e^{i(m+2)\varphi}\sin^{m+2}\theta\frac{d}{d\cos\theta}\left[\sin^{-m-1}\theta\cdot\sin^{m+1}\theta\frac{d}{d\cos\theta}[\sin^{-m}\theta f(\theta)]\right]$$

$$= (-\hbar)^2 e^{i(m+2)\varphi}\sin^{m+2}\theta\left[\frac{d}{d\cos\theta}\right]^2[\sin^{-m}\theta f(\theta)]$$

$$L_+^3[e^{im\varphi}f(\theta)] = \cdots$$

$$= (-\hbar)^3 e^{i(m+3)\varphi}\sin^{m+3}\theta\left[\frac{d}{d\cos\theta}\right]^3[\sin^{m}\theta f(\theta)]]$$

k回 くりかえして

$$L_+^k[e^{im\varphi}f(\theta)] = (-\hbar)^k e^{i(m+k)\varphi}\sin^{m+k}\theta\left[\frac{d}{d\cos\theta}\right]^k[\sin^{-m}\theta f(\theta)]$$

$$L_-[e^{im\varphi}f(\theta)] = \hbar e^{i(m-1)\varphi}\sin^{-(m-1)}\theta\,\frac{d}{d\cos\theta}[\sin^m\theta\, f(\theta)]$$

くりかえして

$$L_-^2[e^{im\varphi}f(\theta)] = \hbar L_-[e^{i(m-1)\varphi}\sin^{-(m-1)}\theta\,\frac{d}{d\cos\theta}[\sin^m\theta\, f(\theta)]]$$

$$= \hbar^2 e^{i(m-2)\varphi}\sin^{-(m-2)}\theta\,\frac{d}{d\cos\theta}[\sin^{m-1}\theta\,\sin^{-(m-1)}\theta\,\frac{d}{d\cos\theta}[\sin^m\theta\, f(\theta)]]$$

$$= \hbar^2 e^{i(m-2)\varphi}\sin^{-(m-2)}\theta\,\left[\frac{d}{d\cos\theta}\right]^2[\sin^m\theta\, f(\theta)]$$

$$L_-^3[e^{im\varphi}f(\theta)] = \cdots$$

$$= \hbar^3 e^{i(m-3)\varphi}\sin^{-(m-3)}\theta\,\left[\frac{d}{d\cos\theta}\right]^3[\sin^m\theta\, f(\theta)]$$

k回くりかえして

$$L_-^k[e^{im\varphi}f(\theta)] = \hbar^k e^{i(m-k)\varphi}\sin^{-(m-k)}\theta\,\left[\frac{d}{d\cos\theta}\right]^k[\sin^m\theta\, f(\theta)]$$

まとめて

$$\begin{cases} L_+^k[e^{im\varphi}f(\theta)] = (-\hbar)^k e^{i(m+k)\varphi}\sin^{m+k}\theta\,\left[\frac{d}{d\cos\theta}\right]^k[\sin^{-m}\theta\, f(\theta)] \\ L_-^k[e^{im\varphi}f(\theta)] = \hbar^k e^{i(m-k)\varphi}\sin^{-(m-k)}\theta\,\left[\frac{d}{d\cos\theta}\right]^k[\sin^m\theta\, f(\theta)] \end{cases}$$

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$$\nabla = \mathbf{e}_r \partial_r + \mathbf{e}_\theta \frac{1}{r}\partial_\theta + \mathbf{e}_\varphi \frac{1}{r\sin\theta}\partial_\varphi$$

$\mathbf{r} = r\mathbf{e}_r$

$\mathbf{L} = \mathbf{r}\times\mathbf{p} = -i\hbar r \mathbf{e}_r \times \nabla = -i\hbar\left(\mathbf{e}_\varphi \partial_\theta - \mathbf{e}_\theta \frac{1}{\sin\theta}\partial_\varphi\right)$ ← rによらない 球面上の関数 $\Omega = (\theta, \varphi)$

$\mathbf{e}_r \times \mathbf{e}_r = 0 \quad \mathbf{e}_r \times \mathbf{e}_\theta = \mathbf{e}_\varphi$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \hbar \begin{pmatrix} i\sin\varphi \\ -i\cos\varphi \\ 0 \end{pmatrix} \partial_\theta + \hbar \begin{pmatrix} i\cos\varphi\cot\theta \\ i\sin\varphi\cot\theta \\ -i \end{pmatrix} \partial_\varphi$$

$ie^{i\varphi} = i(\cos\varphi + i\sin\varphi)$

$$L_x + iL_y = L_+ = \hbar(\underbrace{i\sin\varphi + \cos\varphi}_{e^{i\varphi}})\partial_\theta + (i\cos\varphi - \sin\varphi)\cot\theta\,\partial_\varphi$$
$$= \hbar e^{i\varphi}(\partial_\theta + i\cot\theta\,\partial_\varphi)$$

$$L_x - iL_y = L_- = \hbar(i\sin\varphi - \cos\varphi)\partial_\theta + (i\cos\varphi + \sin\varphi)\cot\theta\,\partial_\varphi$$
$$= \hbar e^{-i\varphi}(-\partial_\theta + i\cot\theta\,\partial_\varphi)$$

$-(\cos\varphi - i\sin\varphi) = -e^{-i\varphi}$

$$L_z = -i\hbar\partial_\varphi$$
$$(\partial^\dagger = -\partial)$$

球面調和関数 $\quad Y_{lm}(\Omega) \quad \Omega = (\theta, \varphi)$

$[L^2, L_z] = 0 \qquad L^2 Y_{lm} = \hbar^2 l(l+1) Y_{lm}$

→ 同時固有状態 $\qquad L_z Y_{lm} = \hbar m Y_{lm}$

$Y_{lm}(\Omega) \qquad L_+ Y_{ll} = L_- Y_{l-l} = 0 \qquad L_z = -i\hbar\partial_\varphi$

$Y_{lm}(\Omega) = \Theta_{lm}(\theta)\Phi_m(\varphi)$ と変数分離形におくと $L_z Y_{lm} = \hbar m Y_{lm}$ より

$$\Phi_m(\varphi) = \frac{1}{\sqrt{2\pi}} e^{im\varphi}$$

ただし $\int_0^{2\pi} d\varphi |\Phi_m(\varphi)|^2 = 1$ と規格化した。

また この関数の一価性より $e^{i2\pi m} = 1$ つまり mは整数である。

$\varphi = 0$と$\varphi = 2\pi$は同じ $(e^{im0} = 1 = e^{i2\pi m})$

よって lも0以上の整数となる $(m = -l \cdots l)$