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Date

2.3.1 極座標での角運動量演算子

Class 7

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \cos\phi \sin\theta \\ r \sin\phi \sin\theta \\ r \cos\theta \end{pmatrix}$$

$$\mathbf{T} = \mathbf{x} \mathbf{D}^{-1} \Rightarrow \mathbf{x} = \mathbf{T} \mathbf{D}$$

$$\mathbf{T} \in \mathbb{R}^3, \quad \tilde{\mathbf{x}} = \tilde{\mathbf{D}} \tilde{\mathbf{T}} \\ \mathbf{T} = \mathbf{D}^{-1}$$

$$(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi) \equiv \left(\frac{\partial \mathbf{r}}{\partial r}, \frac{\partial \mathbf{r}}{\partial \theta}, \frac{\partial \mathbf{r}}{\partial \phi} \right) = \begin{pmatrix} \frac{1}{hr} \frac{\partial x}{\partial r} & \frac{1}{h_\theta} \frac{\partial x}{\partial \theta} & \frac{1}{h_\phi} \frac{\partial x}{\partial \phi} \\ \frac{1}{hr} \frac{\partial y}{\partial r} & \frac{1}{h_\theta} \frac{\partial y}{\partial \theta} & \frac{1}{h_\phi} \frac{\partial y}{\partial \phi} \\ \frac{1}{hr} \frac{\partial z}{\partial r} & \frac{1}{h_\theta} \frac{\partial z}{\partial \theta} & \frac{1}{h_\phi} \frac{\partial z}{\partial \phi} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{pmatrix} \mathbf{D}^{-1} = \begin{pmatrix} \cos\theta \sin\phi & \cos\theta \cos\phi & -\sin\theta \\ \sin\phi \sin\theta & \cos\phi \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} = \mathbf{T}$$

$$\mathbf{D} = \text{diag}(h_r, h_\theta, h_\phi) = \text{diag} \left(\underbrace{1}_{h_r}, \underbrace{r}_{h_\theta}, \underbrace{r \sin\theta}_{h_\phi} \right)$$

$$\begin{aligned} \nabla \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} h_r \tilde{\mathbf{e}}_r \\ h_\theta \tilde{\mathbf{e}}_\theta \\ h_\phi \tilde{\mathbf{e}}_\phi \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= \text{diag}(h_r, h_\theta, h_\phi) \begin{pmatrix} \tilde{\mathbf{e}}_r \\ \tilde{\mathbf{e}}_\theta \\ \tilde{\mathbf{e}}_\phi \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \text{diag}(h_r, h_\theta, h_\phi) \tilde{\mathbf{T}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= \mathbf{D} \tilde{\mathbf{T}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \end{aligned}$$

逆行列 $\tilde{\mathbf{T}}^{-1} = \mathbf{T}$

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = \mathbf{T} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r} \frac{\partial}{\partial \theta} \\ \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} \end{pmatrix} = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \mathbf{e}_\phi \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi}$$

$$\mathbf{r} = r \mathbf{e}_r$$

Class 7

$$\mathbb{L} = \mathbf{r} \times \mathbf{p} = -i\hbar \mathbf{r} \times \nabla = -i\hbar (e^{\phi} \partial_{\theta} - e^{\theta} \frac{1}{\sin\theta} \partial_{\phi})$$

$$\frac{1}{r} e^{\theta} \partial_r + e^{\theta} \frac{1}{r} \partial_{\theta} + e^{\phi} \frac{1}{r \sin\theta} \partial_{\phi}$$

$$= \hbar \begin{pmatrix} i \sin\phi \\ -i \cos\phi \\ 0 \end{pmatrix} \partial_{\theta} + \hbar \begin{pmatrix} i \cos\phi \cot\theta \\ i \sin\phi \cot\theta \\ -i \end{pmatrix} \partial_{\phi} \quad \leftarrow r = \frac{1}{\sin\theta}$$

$$L_{+} = L_x + iL_y = \hbar (i \sin\phi + \cos\phi) \partial_{\theta} + (i \cos\phi - \sin\phi) \cot\theta \partial_{\phi} \cdot \hbar$$

$$= \hbar e^{i\phi} \partial_{\theta} + i e^{i\phi} \cot\theta \partial_{\phi} \cdot \hbar$$

$$= \hbar e^{i\phi} (\partial_{\theta} + i \cot\theta \partial_{\phi})$$

$$L_{-} = \hbar (i \sin\phi - \cos\phi) \partial_{\theta} + (i \cos\phi + \sin\phi) \cot\theta \partial_{\phi} \cdot \hbar$$

$$= \hbar -e^{-i\phi} \partial_{\theta} + i e^{-i\phi} \cot\theta \partial_{\phi} \cdot \hbar$$

$$= \hbar e^{-i\phi} (-\partial_{\theta} + i \cot\theta \partial_{\phi})$$

$$L_z = -i\hbar \partial_{\phi} \quad (\partial^{\dagger} = -\partial)$$

○ 一般論に従い、次の関係を満たす関数を求める

$$\mathbb{L}^2 Y_{jm} = \hbar^2 j(j+1) Y_{jm}, \quad L_z Y_{jm} = \hbar m Y_{jm}, \quad L_{+} Y_{j\ell} = L_{-} Y_{j\ell} = 0$$

$$Y_{jm}(\Omega = (\theta, \phi)) = \Theta_{jm}(\theta) \Phi_m(\phi) \text{ と変数分離形 (} \partial_{\theta} Y_{jm} = \hbar m Y_{jm} \text{)} \quad \leftarrow -i\hbar \partial_{\phi}$$

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi} \quad \left(\int_0^{2\pi} d\phi |\Phi_m(\phi)|^2 = 1 \text{ の規格化} \right)$$

$$\text{関数の一価性より } e^{i2\pi m} = 1 \text{ である } m \text{ は整数 } (e^{i0} = 1 = e^{i2\pi m})$$

よって、 0 以上の整数

$$-l \leq m \leq l$$

$\therefore \tau$ の関数 $f(\theta)$ に対して

$$L_{+} [e^{im\phi} f(\theta)] = e^{im\phi} \hbar e^{i\phi} \left[\frac{df}{d\theta} - m f \cot\theta \right]$$

$$L_- [e^{im\phi} f(\theta)] = -e^{im\phi} h e^{-i\phi} \left[\frac{df}{d\theta} + mf \cos\theta \right]$$

これは以下のように書き直す

$$L_+ [e^{im\phi} f(\theta)] = -h e^{i(m+1)\phi} \sin^{m+1}\theta \frac{d}{d\cos\theta} [\sin^{-m}\theta f(\theta)]$$

$$L_- [e^{im\phi} f(\theta)] = h e^{i(m-1)\phi} \sin^{-(m-1)}\theta \frac{d}{d\cos\theta} [\sin^m\theta f(\theta)]$$

これを k 回 ($k < m$) 繰り返す

$$L_+^k [\dots] = (-h)^k e^{i(m+k)\phi} \sin^{m+k}\theta \left[\frac{d}{d\cos\theta} \right]^k [\sin^{-m}\theta f(\theta)]$$

$$L_-^k [\dots] = h^k e^{i(m-k)\phi} \sin^{-(m-k)}\theta \left[\frac{d}{d\cos\theta} \right]^k [\sin^m\theta f(\theta)]$$