5/12　第4回　　2017I0848 有富 洸人

解析力学

- motion eq.　　$F = m\ddot{x}$
- 保存力　　$F = -\partial_x V$

1. 初期値問題

x_0, v_0

2. 最小作用の原理

$$S[x(t)] \equiv \int_{t_i}^{t_f} dt\, L(x(t), \dot{x}(t))$$

$\rightarrow \delta S = 0$ となる

$$\Rightarrow \frac{\delta L}{\delta x} \equiv \frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \quad (\text{Euler-Lagrange eq.})$$

$F = -\partial_x V$ のとき　$L = \frac{1}{2} m\dot{x}^2 - V$ とする.

$$\rightarrow \frac{\delta L}{\delta x} = -\partial_x V - \frac{d}{dt} m\dot{x} = \underbrace{F - m\ddot{x} = 0}_{\text{Newton}}$$

正準方程式へ

$L(x, \dot{x}) \rightarrow H(x, p) \qquad \left(p \equiv \frac{\partial L}{\partial \dot{x}}\right)$

Legendre 変換　$H \equiv \dot{x} p - L$

$$\begin{aligned} \rightarrow \delta H &= \delta \dot{x} p + \dot{x} \delta p - \delta L \\ &= \delta \dot{x} p + \dot{x} \delta p - \frac{\partial L}{\partial x} \delta x - \frac{\partial L}{\partial \dot{x}} \delta \dot{x} \\ &= \delta \dot{x} \left(p - \frac{\partial L}{\partial \dot{x}}\right) + \dot{x} \delta p - \frac{\partial L}{\partial x} \delta x \\ &= \dot{x} \delta p - \frac{\partial L}{\partial x} \delta x \end{aligned}$$

$\rightarrow H$ は $\delta p, \delta x$ の関数

- 正準方程式　$\frac{\partial H}{\partial p} = \dot{x}, \quad \frac{\partial H}{\partial x} = -\frac{\partial L}{\partial x} = -\dot{p}$

$$\begin{aligned} \rightarrow H &= \dot{x} p - L = m \dot{x}^2 - \left(\frac{1}{2} m \dot{x}^2 - V\right) \\ &= \frac{p^2}{2m} + V \end{aligned}$$

$$\rightarrow \frac{\partial H}{\partial p} = \frac{p}{m} = \dot{x}, \quad \frac{\partial H}{\partial x} = \partial_x V = -\dot{p}$$

$\therefore$ $m \ddot{x} = -\partial_x V = F$

Newton eq.

ネーターの定理

・空間並進

$x \mapsto x' = x + \delta a$ と変換

$$L'(x', \dot{x}') = L(x, \dot{x})$$

$$S = \int_{t_i}^{t_f} dt\ L(x, \dot{x}) = \int_{t_i}^{t_f} dt\ L'(x', \dot{x}')$$

$$0 = \int_{t_i}^{t_f} dt \left\{ L'(x', \dot{x}') - L(x, \dot{x}) \right\}$$

↓ L 不変

$$= \int_{t_i}^{t_f} dt \left\{ \underbrace{L(x', \dot{x}') - L(x, \dot{x})}_{\delta L} \right\}$$

$$= \int_{t_i}^{t_f} dt\ \delta L$$

$$= \int_{t_i}^{t_f} dt \left\{ \frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \delta \dot{x} \right\}$$

$$= \int_{t_i}^{t_f} dt \left\{ \frac{\partial L}{\partial x} \delta x + \frac{d}{dt}\left(\delta x \frac{\partial L}{\partial \dot{x}} \right) - \delta x \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right\}$$

$$= \int_{t_i}^{t_f} dt \left\{ \frac{d}{dt}\underbrace{\left(\delta x \frac{\partial L}{\partial \dot{x}} \right)}_{G} + \frac{\delta L}{\delta x}^{0} \delta x \right\}$$

$$= \int_{t_i}^{t_f} dt \cdot \frac{d}{dt} G = G \int_{t_i}^{t_f} = 0$$

$G = \delta x \frac{\partial L}{\partial \dot{x}} = \delta a \frac{\partial L}{\partial \dot{x}}$ が保存量

$\rightarrow \frac{\partial L}{\partial \dot{x}} = p$: 運動量

$\rightarrow$ 空間並進では運動量が保存する.

- 空間回転

$\mathbf{r} \mapsto \mathbf{r}' = R\mathbf{r}$, $\tilde{R}R = E_3$

$L'(\mathbf{r}', \dot{\mathbf{r}}') = L(\mathbf{r}, \dot{\mathbf{r}})$

L の回転不変性 : $\begin{cases} L'(\mathbf{r}', \dot{\mathbf{r}}') = L(\mathbf{r}, \dot{\mathbf{r}}) \\ L'(\mathbf{r}, \dot{\mathbf{r}}) = L(\mathbf{r}, \dot{\mathbf{r}}) \end{cases}$

$$S = \int_{t_i}^{t_f} dt\, L'(\mathbf{r}', \dot{\mathbf{r}}') = \int_{t_i}^{t_f} dt\, L(\mathbf{r}, \dot{\mathbf{r}})$$

$$0 = \int_{t_i}^{t_f} dt \left\{ L'(\mathbf{r}', \dot{\mathbf{r}}') - L(\mathbf{r}, \dot{\mathbf{r}}) \right\}$$

$$= \int_{t_i}^{t_f} dt \left\{ L(\mathbf{r}', \dot{\mathbf{r}}') - L(\mathbf{r}, \dot{\mathbf{r}}) \right\}$$

↓無限小変換

$$= \int_{t_i}^{t_f} dt \left\{ L(\mathbf{r} + \delta\mathbf{r}, \dot{\mathbf{r}} + \delta\dot{\mathbf{r}}) - L(\mathbf{r}, \dot{\mathbf{r}}) \right\}$$

$$= \int_{t_i}^{t_f} dt\, \delta L$$

$$= \int_{t_i}^{t_f} dt \left\{ \delta r_i \frac{\partial L}{\partial r_i} + \delta \dot{r}_i \frac{\partial L}{\partial \dot{r}_i} \right\}$$

$$= \int_{t_i}^{t_f} dt \left\{ \frac{d}{dt} G + \delta r_i \frac{\delta L}{\delta r_i} \right\} = 0$$

↙ Euler-Lagrange eq.

$$= \int_{t_i}^{t_f} dt \frac{d}{dt} G$$

$$= G|_{t_f} - G|_{t_i} = 0$$

→ G が保存

$$G = \delta r_i \frac{\partial L}{\partial \dot{r}_i} \quad , \quad p_i = \frac{\partial L}{\partial \dot{r}_i}$$

$$= \delta \mathbf{r} \cdot \mathbf{p}$$

$$= (\delta \boldsymbol{\omega} \times \mathbf{r}) \cdot \mathbf{p}$$

$$= (\mathbf{r} \times \mathbf{p}) \cdot \delta \boldsymbol{\omega}$$

$$= \mathbf{L} \cdot \delta \boldsymbol{\omega} \quad \text{が保存.}$$

→ $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ ：角運動量が保存.

a　時間推進

$$t \to t'$$
$$q(t) \to q'(t')$$
$$\dot{q}(t) \to \dot{q}'(t')$$

$$\to \quad L'(q', \dot{q}', t')dt' = L(q, \dot{q}, t)\,dt$$

$$S = \int_{t_i}^{t_f} dt\, L(q(t), \dot{q}(t), t)$$

$$= \int_{t_i'}^{t_f'} dt'\, L'(q'(t'), \dot{q}'(t'), t')$$

$$= \int_{t_i'}^{t_f'} dt\, L'(q'(t), \dot{q}'(t), t)$$

Lの不変性 ： $L'(q', \dot{q}', t) = L(q', \dot{q}', t')$

$$\to \int_{t_i}^{t_f} dt\, L(q, \dot{q}, t) = \int_{t_i'}^{t_f'} dt\, L(q', \dot{q}', t)$$

$$\delta t_f\, L\big|_{t_f} - \delta t_i L\big|_{t_i} + \int_{t_i}^{t_f} dt\, \delta L = 0$$

$$0 = \int_{t_i}^{t_f} dt \left[\frac{d}{dt}\left\{ \delta t L + \delta q \frac{\partial L}{\partial \dot{q}} \right\} + \delta q \frac{\delta L}{\delta q} \right]$$

↙ Newton eq.

$$= \int_{t_i}^{t_f} dt\, \frac{d}{dt} G$$

$$= G\big|_{t_f} - G\big|_{t_i}$$

→ G が保存. $\quad G = \delta t\, L + \delta q \dfrac{\partial L}{\partial \dot{q}}$

$$\to G = \delta t\, L + (-\delta t\, \dot{q}) \frac{\partial L}{\partial \dot{q}}$$

$$= \delta t \left\{ L - \dot{q} \frac{\partial L}{\partial \dot{q}} \right\}$$

$$= \delta t \left\{ L - \dot{q} p \right\}$$

$$= -\delta t\, H \qquad \text{が保存.}$$

→ エネルギー保存則.

- Lagrangian の不変性 → 保存則

  ⇒ 量子論と整合的.