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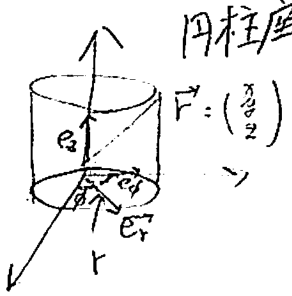
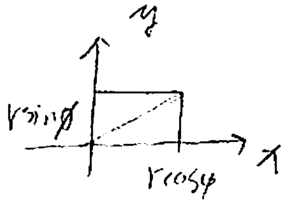
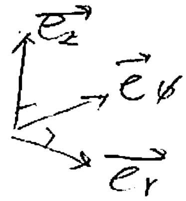
座標変換の例

円柱座標（2次の極座標）





$x = r\cos\phi$
$y = r\sin\phi$
$z = z$



$$\vec{e}_r = \widehat{\frac{\partial \vec{r}}{\partial r}} = \frac{\frac{\partial \vec{r}}{\partial r}}{\left|\frac{\partial \vec{r}}{\partial r}\right|}$$

$$\vec{r} = \vec{e}_1 x + \vec{e}_2 y + \vec{e}_3 z$$

$$= (\vec{e}_1, \vec{e}_2, \vec{e}_3)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (\vec{e}_1, \vec{e}_2, \vec{e}_3)\begin{pmatrix} r\cos\phi \\ r\sin\phi \\ z \end{pmatrix}$$

$$\frac{\partial \vec{r}}{\partial r} = (\vec{e}_1, \vec{e}_2, \vec{e}_3)\begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix}$$

$$\left|\frac{\partial \vec{r}}{\partial r}\right|^2 = \cos^2\phi + \sin^2\phi + 0 = 1$$

$$\vec{e}_r = (\vec{e}_1, \vec{e}_2, \vec{e}_3)\begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix} = \theta e_r \qquad e_r = \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix}$$

$$\vec{e}_\phi = \widehat{\frac{\partial \vec{r}}{\partial \phi}} = \frac{\frac{\partial \vec{r}}{\partial \phi}}{\left|\frac{\partial \vec{r}}{\partial \phi}\right|} \qquad \frac{\partial \vec{r}}{\partial \phi} = (\vec{e}_1, \vec{e}_2, \vec{e}_3)\begin{pmatrix} -r\sin\phi \\ r\cos\phi \\ 0 \end{pmatrix}$$

$$\left|\frac{\partial \vec{r}}{\partial \phi}\right|^2 = \left|\frac{\partial r}{\partial \phi}\right|^2 = r^2\sin^2\phi + r^2\cos^2\phi + C = r^2 \qquad \left|\frac{\partial \vec{r}}{\partial \phi}\right| = r$$

$$\vec{e}_\phi = (\vec{e}_1, \vec{e}_2, \vec{e}_3)\begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}, \quad e_\phi = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix} = \theta e_\phi$$

$$\vec{e}_z = \vec{e}_3 = (\vec{e}_1, \vec{e}_2, \vec{e}_3)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \theta e_z \qquad e_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$(\mathbb{A}\times\mathbb{B})\cdot(\mathbb{C}\times\mathbb{D}) = (\mathbb{A}\cdot C)(\mathbb{B}\cdot D) - (\mathbb{A}\cdot\mathbb{D})(\mathbb{B}\cdot\mathbb{C})$ ＊

$(\mathbb{A}\times\mathbb{B})_i = \varepsilon_{iab}A_aB_b$

$(\mathbb{C}\times\mathbb{D})_i = \varepsilon_{icd}C_cD_d$

$$(\mathbb{A}\times\mathbb{B})\cdot(\mathbb{C}\times\mathbb{D}) = (\mathbb{A}\times\mathbb{B})_i(\mathbb{C}\times\mathbb{D})_i = \varepsilon_{iab}A_aB_b\,\varepsilon_{icd}C_cD_d$$
$$= \varepsilon_{iab}\varepsilon_{icd}A_aB_bC_cD_d$$

$(\mathbb{A}\cdot\mathbb{C})(\mathbb{B}\cdot\mathbb{D}) = \delta_{ac}A_aC_c\ \delta_{bd}B_bD_d = (\delta_{ac}\delta_{bd})A_aB_bC_cD_d$

$(\mathbb{A}\cdot\mathbb{D})(\mathbb{B}\cdot\mathbb{C}) = \delta_{ad}A_aD_d\,\delta_{bc}B_bC_c = (\delta_{ad}\delta_{bc})A_aB_bC_cD_d$

＊より $\varepsilon_{iab}\varepsilon_{icd}A_aB_bC_cD_d = (\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})A_aB_bC_cD_d$

ここで $A_a\ B_b\ C_c\ D_d$ 任意

$\Rightarrow \varepsilon_{iab}\varepsilon_{icd} = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}$ 縮約

一般に $|\vec{A}\times\vec{B}|^2 = |\mathbb{A}\times\mathbb{B}|^2 = (\mathbb{A}\times\mathbb{B})\cdot(\mathbb{A}\cdot B)$

$= \varepsilon_{iab}A_aB_b\varepsilon_{icd}A_cB_d$

$= \varepsilon_{iab}\varepsilon_{icd}A_aB_bA_cB_d$

$= (\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})A_aB_bA_cB_d$ $\quad AB\cos\theta$

$= A_aB_bA_aB_b - \boxed{A_aB_bA_bB_a}$

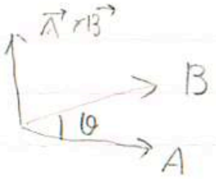
$= |\mathbb{A}|^2|\mathbb{B}|^2 - (\mathbb{A}\cdot\mathbb{B})^2\ \ |\mathbb{A}|^2|\mathbb{B}|^2\cos^2\theta$

$= |\mathbb{A}|^2|\mathbb{B}|^2(1-\cos^2\theta) = |\mathbb{A}|^2|\mathbb{B}|^2\sin^2\theta$ $\quad$ AとBのなす角θ

$|\vec{A}\times\vec{B}| = |\vec{A}||\vec{B}|\sin\theta$

$\vec{A}\parallel\vec{B} \to \theta = 0, \pi \Rightarrow \vec{A}\times\vec{B} = 0$

$\vec{e}_1\times\vec{e}_2\times\vec{e}_3$ などを考えて $\vec{A}\times\vec{B}$：$\vec{A}\to\vec{B}$ へねじをまわすときねじの進む方向



$(\mathbb{A}\times\mathbb{B})\cdot\mathbb{A} = (\varepsilon_{ijk}A_jB_k)A_i = \varepsilon_{ijk}A_iA_jB_k$

$= \frac{1}{2}\varepsilon_{ijk}(A_iA_j + A_jA_i)B_k = \frac{1}{2}\varepsilon_{ijk}A_iA_jB_k + \frac{1}{2}\varepsilon_{ijk}A_jA_iB_k$

$= \frac{1}{2}(\varepsilon_{ijk} + \varepsilon_{jik})A_iA_jB_k$

＊より

$\varepsilon_{iab}\varepsilon_{icd}A_aB_bC_cD_d = (\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})A_aB_bC_cD_d$

$= (\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})A_aB_bC_cD_d$

ここで $A_a\ B_b\ C_c\ D_d$ 任意

一般にベクトル$\vec{v}$

$$\vec{v} = \vec{e_1}v_1 + \vec{e_2}v_2 + \vec{e_3}v_3 = (\vec{e_1}, \vec{e_2}, \vec{e_3})\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \theta v$$

$$= (\vec{e_r}, \vec{e_\phi}, \vec{e_z})\begin{pmatrix} v_r \\ v_\phi \\ v_z \end{pmatrix} \qquad \| \quad \theta T_{r\phi z}\begin{pmatrix} v_r \\ v_\phi \\ v_z \end{pmatrix}$$

$\begin{pmatrix} v_r \\ v_\phi \\ v_z \end{pmatrix}$ 円柱座標での成分

3×3行列

$$(\vec{e_r}, \vec{e_\phi}, \vec{e_z}) = (\vec{e_1}, \vec{e_2}, \vec{e_3})(\cancel{e_1}, \cancel{e_2}, \cancel{e_3}) = \theta T_{r\phi z},$$

$$T_{r\phi z} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \tilde{T}T = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$v = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = T_{r\phi z}\begin{pmatrix} v_r \\ v_\phi \\ v_z \end{pmatrix}, \quad \tilde{T} = T^{-1}$$

$$\begin{pmatrix} v_r \\ v_\phi \\ v_z \end{pmatrix} = \tilde{T}_{r\phi z}\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$\vec{r}$に質点が存在している時の速度ベクトル$\vec{v}$，加速度ベクトル$\vec{a}$
は円柱座標系$\theta_{r\phi z} = (\vec{e_r}, \vec{e_\phi}, \vec{e_z})$ でどう表わせるか

以上は$\vec{v}$：一般のベクトル

$$\vec{r} = \vec{e_1}x + \vec{e_2}y + \vec{e_3}z = \theta r$$

$$\vec{v} = \dot{\vec{r}} = \theta \dot{r} = \theta v$$

← r, φは時間tの関数

$$v = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} \quad \begin{matrix} x = r\cos\phi \\ y = r\sin\phi \\ z \quad z \end{matrix} \leftarrow x = x(t) = r(t)\cos\phi(t)$$

$$\dot{x} = \frac{d}{dt}x(t) = \dot{r}\cos\phi - r\dot{\phi}\sin\phi$$

$\dot{y} = \dot{r}\sin\phi + r\dot{\phi}\cos\phi$　よって一般のベクトルの変換則(※※より)

$$\begin{pmatrix} V_r \\ V_\phi \\ V_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{r}\cos\phi - r\dot{\phi}\sin\phi \\ \dot{r}\sin\phi + r\dot{\phi}\cos\phi \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \dot{r}\cos^2\phi - r\dot{\phi}\sin\phi\cos\phi + \dot{r}\sin^2\phi + r\dot{\phi}\sin\phi\cos\phi \\ -\dot{r}\sin\phi\cos\phi + r\dot{\phi}\sin^2\phi + \dot{r}\sin\phi\cos\phi + r\dot{\phi}\cos^2\phi \\ \dot{z} \end{pmatrix}$$

$$\vec{V} = (\vec{e_r}, \vec{e_\phi}, \vec{e_z}) \begin{pmatrix} \dot{r} \\ r\dot{\phi} \\ \dot{z} \end{pmatrix}$$

加速度ベクトル

$$\vec{a} = \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = \begin{pmatrix} \ddot{r}\cos\phi - \dot{r}\dot{\phi}\sin\phi - \dot{r}\dot{\phi}\sin\phi - r\ddot{\phi} - r\dot{\phi}\dot{\phi}\cos\phi \\ \ddot{r}\sin\phi + \dot{r}\dot{\phi}\cos\phi + \dot{r}\dot{\phi}\cos\phi + r\ddot{\phi}\cos\phi - r\dot{\phi}\dot{\phi}\sin\phi \\ \ddot{z} \end{pmatrix}$$

$= (\ddot{r} - r\dot{\phi}^2)\cos\phi - (2\dot{r}\dot{\phi} + r\ddot{\phi})\sin\phi$

$= (2\dot{r}\dot{\phi} + r\ddot{\phi})\cos\phi + (\ddot{r} - r\dot{\phi}^2)\sin\phi$

$\vec{a} = \Theta a$

$$a = \begin{pmatrix} (\ddot{r} - r\dot{\phi}^2)\cos\phi - (2\dot{r}\dot{\phi} + r\ddot{\phi})\sin\phi \\ (\ddot{r} - r\dot{\phi}^2)\sin\phi + (2\dot{r}\dot{\phi} + r\ddot{\phi})\cos\phi \\ \ddot{z} \end{pmatrix}$$

※※より

$$\begin{pmatrix} a_r \\ a_\phi \\ a_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \\ \\ \\ \end{pmatrix} = \begin{pmatrix} \ddot{r} - r\dot{\phi}^2 \\ 2\dot{r}\dot{\phi} + r\ddot{\phi} \\ \ddot{z} \end{pmatrix}$$

$$\vec{a} = (\vec{e_r}, \vec{e_\phi}, \vec{e_z}) \begin{pmatrix} \ddot{r} - r\dot{\phi}^2 \\ 2\dot{r}\dot{\phi} + r\ddot{\phi} \\ \ddot{z} \end{pmatrix} \Rightarrow \mathbb{F} = (\vec{e_r}, \vec{e_\phi}, \vec{e_z}) \begin{pmatrix} F_r \\ F_\phi \\ F_z \end{pmatrix}$$ とすれば

$F_r = m(\ddot{r} - r\dot{\phi}^2)$　$F_\phi = m(2\dot{r}\dot{\phi} + r\ddot{\phi})$　$F_z = m\ddot{z}$

円柱座標系での運動方程式