Topological identification by quantized Berry phases: spin-singlet pairs and edge states

Isao Maruyama
Osaka University, Japan

S.Tanaya, M.Arikawa, Y.Hatsugai
Tsukuba University, Japan
How to identify Phase diagram

To identify a phase diagram,

- Energy, Entropy, Excitation Gap
- Order Parameter (magnetization,...)
- Correlation function
- Non-local order parameter
- Quantum Entanglement
- Topological quantity

Phase diagram of a spin ladder
How to identify Phase diagram

To identify a phase diagram,

- Energy, Entropy, Excitation Gap
- Order Parameter (magnetization, ...)
- Correlation function
- Non-local order parameter
  Liep-Shultz-Mattis twist operator
- Quantum Entanglement
  Entanglement Concurrence
- Topological quantity
  Quantized Berry Phase

Our study - PRB.79.115107

J.L.Song, et.al, PRB.74.155119
See also, E. H. Kim, et.al, PRB.77.205121
Introduction of
Quantized Berry phase

- Topological quantity as order parameters
- Chern Number in QH system, (intrinsically integer)
- Quantized Berry phase, (quantized due to a symmetry)
Introduction of Quantized Berry phase

- Topological quantity as order parameters
- Chern Number in QH system, (intrinsically integer)
- Quantized Berry phase, (quantized due to a symmetry)

1. Quantized Berry phase of a multiplet below a gap is proposed to characterize a topological or quantum order.
   Y. Hatsugai, JPSJ.75.123601
   → Stable against the perturbation due to the gap

2. Quantized Berry phase of the ground state without a gap
   Aligia, EuroPhysLett.45.411
Definition of Berry’s phase

Given Hamiltonian $H$

1. Make one parameter dependent Hamiltonian $H(t)$ with the periodicity $H(0)=H(2\pi)$

2. Adiabatic time evolution $|gs(t)>$ via $H(t)$
   ~ "Adiabatic" means $H(t)|gs(t)>=E(t)|gs(t)>$ ~

3. $|gs(0)> = |gs(2\pi)>$?
   ~ No! Phase factor can be different. ~

$|gs(2\pi)> = \exp(i\gamma) |gs(0)>$, $\gamma$: Berry Phase

Definition of Berry phase

- **Berry phase for** $M$ **states**
  \[ \gamma = -\sum_{m=1}^{M} \text{Im} \int_{C} \langle \psi_m(\theta) | d\psi_m(\theta) \rangle \]
  \[ H(\theta)|\psi_m(\theta)\rangle = E_m(\theta)|\psi_m(\theta)\rangle \]
  \[ (M=1 \text{ for unique ground state}) \]

  *Wilczek, Zee* PRL.52.2111 (1984)

- **“Lattice” Berry phase**
  \[ \gamma = -\arg \Gamma_N \]
  \[ \Gamma_N = \det \prod_{i=1}^{N} C(\theta_i) \]
  \[ (C(\theta_i))_{mn} = \langle \psi_m(\theta_i) | d\psi_n(\theta_{i+1}) \rangle \]
  \[ |\psi_m(\theta_{N+1})\rangle := |\psi_m(\theta_1)\rangle \]

  *Fukui, Hatsugai, Suzuki* JPSJ.74.1674 (2005)

- **Discretization**
  \[ \theta \rightarrow \theta_i, i = 1 \sim N \]

  *KingSmith, Vanderbilt* PRB.47.1651, (1993)

  *JPSJ.75.123601 Y.Hatsugai*

- **Berry phase is Quantized**
  \[ \gamma = 0, \pi \pmod{2\pi} \]

  \[ [H(\theta), \Theta] = 0 \]

  Anti-Unitary operator \[ \Theta = KU \]
Berry phase applied to a spin system

Hamiltonian $H$ is the Hamiltonian of the spin system.

1. $H(t)$ with the periodicity $H(0)=H(2\pi)$

Local Spin Twist introduces a flux $t$

$$S_i^+ S_j^- + S_i^- S_j^+ \rightarrow e^{i \cdot t} \cdot S_i^+ S_j^- + e^{-i \cdot t} \cdot S_i^- S_j^+$$

Advantages

1. Berry phase is quantized due to the Time Reversal Symmetry. $Z_2$ topological quantity.
2. Simple decoupled model can be obtained by adiabatic modification.
Early studies: Singlet and the Dimerized Heisenberg chain

- Dimerized S=1/2 AF chain

**Strong AF bond**  \[ \cdots \downarrow - \downarrow - \downarrow - \downarrow - \downarrow \cdots \]  **Weak AF bond**

Local Spin Twist introduces a flux $\pi$

\[
S_i^+ S_j^- + S_i^- S_j^+ \rightarrow e^{i\pi} S_i^+ S_j^- - e^{-i\pi} S_i^- S_j^-
\]

If the local spin twist is introduced into a strong bond, We get $\pi$-Berry phase.
Early studies:
Singlet and the Dimerized Heisenberg chain

- Dimerized S=1/2 AF chain

Strong AF bond

Weak AF bond

Local Spin Twist introduces a flux $t$

$$ S_i^+ S_j^- + S_i^- S_j^+ \rightarrow e^{i t} S_i^+ S_j^- - e^{-i t} S_i^- S_j^- $$

If the local spin twist is introduced into this bond, We get Zero-Berry phase.
Then, we leave the link in black.
Early studies:
Singlet and the Dimerized Heisenberg chain

- Dimerized Heisenberg chain

Hamiltonian

Berry phase

Strong AF exchange
Early studies:
Singlet and the Dimerized Heisenberg chain

- Dimerized Heisenberg chain

Hamiltonian

Berry phase

Strong AF exchange

Berry phase of uniform chain is not defined due to a gap closing.

Note that conventional quantity cannot give such clear classification.

- In the strong AF coupling limit, we obtain a decoupled model without changing the Berry phase.

Singlet: \[ \uparrow \downarrow \] gives \( \pi \)-Berry phase
Singlet is detected by the Berry phase

- In the Heisenberg model, $\pi$-bond means “localized singlet”
- In the Kondo insulators, $\pi$-bond means “Kondo singlet”
- In the $t$-$J$ model, Singlet can move as “itinerant singlet”
- In the Spin Ladder, Haldane phase is also identified
- In the *Spin Ladder with ring exchange*, Berry phase identified the Rung singlet, and plaquette singlet
S=1/2 Spin Ladder with four-body ring exchange interaction.

MODEL
$H_{\text{cyc}} = J \left[ \sum_{i=1}^{N/2} \sum_{\alpha=1,2} S_{i,\alpha} \cdot S_{i+1,\alpha} + \sum_{i=1}^{N/2} S_{i,1} \cdot S_{i,2} \right] + K \sum_{i=1}^{N/2} (P_i + P_i^{-1})$

\[
P_i + P_i^{-1} = S_{i,1} \cdot S_{i+1,1} + S_{i,2} \cdot S_{i+1,2} + 4(S_{i,1} \cdot S_{i+1,1})(S_{i,2} \cdot S_{i+1,2}) + S_{i,1} \cdot S_{i,2} + S_{i+1,1} \cdot S_{i+1,2} + 4(S_{i,1} \cdot S_{i,2})(S_{i+1,1} \cdot S_{i+1,2}) + S_{i,1} \cdot S_{i+1,2} + S_{i,2} \cdot S_{i+1,1} - 4(S_{i,1} \cdot S_{i+1,2})(S_{i,2} \cdot S_{1,i+1})
\]

Local spin twist is introduced into leg, rung, and diagonal link.

Frustration
Berry phases of Spin Ladder with ring exchange

Leg Berry phase
Rung Berry phase
Diagonal Berry phase

Result of N=8x2
Berry phases of Spin Ladder with ring exchange

Leg Berry phase
Rung Berry phase
Diagonal Berry phase

Result of N=8x2

\[ \theta = 0.8 \]

\[ \theta = 1.8 \]
Decoupled model

- It has been shown that the Berry phase is useful for a frustrated two-leg ladder to clarify the Rung singlet, Dominant Vector Chirality, and Haldane phase. Maruyama, et.al., AX.0806.4416

- Lieb-Schulz-Mattis twist operator
- Entanglement
- Quantized Berry phase

J.L.Song, et.al, PRB.74.155119, E. H. Kim, et.al, PRB.77.205121

\[ H_{RS} = \sum_{i=1}^{N/2} S_{i,1} \cdot S_{i,2} \]

\[ H_{DVC} = \sum_{i=1}^{N/4} (S_{2i,1} \times S_{2i,2}) \cdot (S_{2i+1,1} \times S_{2i+1,2}) \]

Decoupled model obtained by the Adiabatic transformation
After making a **decoupled** model in each phase, we obtain a trial state.

**TRIAL STATES**
Trial State of dimer covering corresponds to a decoupled model.

Rung Singlet phase

\[ \Psi = \]  

Vector Chirality phase

\[ \Psi = \text{Vector Chirality} + \text{Vector Chirality} \]

These trial states

- predict existence of “Kennedy triplet”
- give a qualitative explanation about Entanglement Entropy
The trial state tells us...

"Kennedy triplet" exists in the gap
The trial state tells us

"Kennedy triplet" exists in the gap
Vector Chirality

\[ \psi = \begin{array}{c}
\text{OBC(Diagonal edge)} \\
(\text{a) Diagonal-edge)}
\end{array} + \begin{array}{c}
\text{OBC(Vertical Edge)} \\
(\text{b) Vertical-edge)}
\end{array} \]

Numerical Results

Kennedy triplet

\[ \Delta E / J = 1.53 \exp(-N / \xi) \quad \xi = 3.22 \]
Trial State

\[ \Psi = \begin{pmatrix} \text{Diagram 1} \\ \text{Diagram 2} \end{pmatrix} + \begin{pmatrix} \text{Diagram 3} \\ \text{Diagram 4} \end{pmatrix} \]

3 log 2 is the exact value of the trial state.
“Existence of Kennedy triplet”

QUANTIZED BERRY PHASE FOR THE OPEN BOUNDARY CONDITION
Quantized Berry Phase (OBCs)

Vector Chirality

OBC(Vertical Edge)

(b) Vertical-edge

OBC(Diagonal edge)

(a) Diagonal-edge

Numerical Results
Quantized Berry Phase with the Open boundary condition

- Dimerized Heisenberg chain (PBC)
  - Strong AF exchange

  Hamiltonian

  Berry phase

- Open Boundary Condition

  Hamiltonian

  Berry phase

  Kennedy Triplet

  Not exist
  Exist
Quantized Berry Phase with the Open boundary condition

- Dimerized Heisenberg chain (PBC)
- Open Boundary Condition

Berry Phase Mismatch

Existence of Kennedy Triplet

Strong AF exchange

Exist
Quantized Berry Phase with the Open boundary condition

- Dimerized Heisenberg chain (PBC)
  - Strong AF exchange

Berry Phase Mismatch

Existence of Kennedy Triplet

This model connects to $S=1$ AF chain.
Vector Chirality Phase

Berry Phase and Kennedy triplet

OBC (vertical edge)

OBC (diagonal edge)
Berry Phase and Kennedy triplet

Berry Phase Mismatch

Existence of Kennedy Triplet

Vector Chirality Phase

PBC

OBC (diagonal edge)
Summary

• It has been shown that the quantized Berry phase is useful to clarify phases of a spin ½ ladder.
  
  I. Maruyama, Y. Hatsugai, PRB. 79.115107

• Adiabatic modification to the decoupled model tells us a simple picture and a trial state.

• The trial states do not show quantitative agreement with energy, and so on. However, the trial states make it possible to understand the numerical results of the topological properties, i.e., the edge states and the entanglement entropy.

  M. Arikawa, S. Tanaya, I. Maruyama, Y. Hatsugai, AX. 0812.3445

• Existence of Kennedy triplet corresponds to quantized Berry phase under OBC.
Rung Singlet Phase

Berry Phase and Kennedy triplet

OBC (vertical edge)

OBC (diagonal edge)

\[
\text{Gap vs. } \frac{1}{N}
\]
end