Condensed matter physics lecture notes

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(Dated: Nov/29/2007)

Part I

Introduction(2007/09/04)

Geometric phase

Quantum liquid

- Quantum Hall system
- Quantum Super conductor
- Kondo insulator
- 1-dimension integer spin chain
- Materials with strong correlation
  Spin systems with frustration
  dimers
- No symmetry breaking (Symmetry breaking is no more important)

What is the Phase ?
Ans) State of matters. In physics, we need quantitative analysis to describe system. To describe phase transition, we use order parameter.

(Ex) Ferromagnets
Let us define local magnetization $\vec{m}(\vec{r})$ as a direction of magnetization at $\vec{r}$. In ferromagnets, $\vec{m}(\vec{r})$ independent $\vec{r}$. Then, the correlation length is infinity. It represents that ferromagnet has long range order. Changing of local order parameter to long range order parameter, one can see the phase change.

Spontaneous Symmetry breaking
Locally, the system has no directional property. But, many particles (size of system $\to \infty$) makes directional properties. (i.e.) Micro system which has symmetry. But sum of micro system, macro system has no more this symmetry. [Spontaneous(it need not existence of external magnetic field) symmetry breaking]
First, let us assume one finite system which has external magnetic field $\vec{B} \neq 0$(There are some special direction.). If one reduce the external magnetic fields, then symmetry will be recover. Let us think two ways of limit,
(a) $\vec{B} \to 0 \Longrightarrow N \to \infty$ [it is not ferromagnets]
(b) $N \to \infty \Longrightarrow \vec{B} \to 0$ [it is a ferromagnet]
(a) and (b) are not equivalent.

Examples of spontaneous symmetry breaking : Ferromagnets, super conductor
Spontaneous symmetry breaking(order phase)
Existence of long distance order is not always same to Spontaneous symmetry breaking.
[Existence of long distance order≈Spontaneous symmetry breaking]

Order parameter gives the phase. Phase transition = phenomena of changing phase.

Classical phase transition : In low temperature, system has order, temperately fluctuation is down. It related competition of entropy and stabilization energy.

Quantum phase transition (QPT): Transition as changing of physical parameter near the zero temperature.

(example of QPT)

Mott transition (Metal-insulator) : In low temperature, changing of electron correlation or doping, metal become insulator! In this case, order parameter is not clear. (Naively electricity conduction)

Transition of plate in the Quantum Hall system :
Hall conduction

$$\sigma_{xy} = (\text{integer}) \times \frac{e^2}{h}$$

In this transition, integer change.

Change of Intensity of disorder , magnetic field (filling)

Quantum liquids
No symmetry breaking
No classical order parameter
(ex) Transition in Quantum Hall system
Non-isotopic superconducts (Breaking of $U(1)$ symmetry) - there are some classes of superconductor.

Geometrical phase(Important part of Quantum effect) $\rightarrow$ Distinguish Quantum liquids

(example of quantum liquids)

Kondo insulator (strong-correlation matter)
Kondo effect : loss of spin by quantum effect.
1-dimension : polyacetylene - strength of bonding is changing [If 1 lattice point has 1 electron, the it become insulator]
2-dimension : graphen - effective mass approximation is no more established. When conduction band and valence band meet one point, energy $E \sim c\left|\vec{k}\right|$, and it become Dirac fermions (it follows Dirac equation).

Part II
Quantum Hall effects (2007/09/11)

(picture) Two dimension electron gas. $I_x = \sigma_{xy} V_y, \sigma_{xy}$:quantize.

$$\begin{aligned}\sigma_{xy} &= \text{integer } \rightarrow \text{integer Q.H.E.(Quantum Hall Effect)} \\ &= p/q \rightarrow \text{fractional Q.H.E.}\end{aligned}$$

$(p.q) = 1$ mutually prime integers. Mostly q is odd. q is even in denominator state.

I. CLASSICAL DESCRIPTION

Charged particles in an electromagnetic filed
Equation of motion ?
Lorentz force is given by

$$\begin{aligned} m\ddot{\vec{r}} &= e(\vec{E} + \vec{v} \times \vec{B}), \\ \text{where, } m &: \text{mass} \\ e &: \text{charge} \\ \vec{B} &: \text{magnetic field} \\ \vec{E} &: \text{Electric field} \\ \vec{r}(t) &: \text{position of particle} \end{aligned}$$

Time t change $t_i \to t_f$. $\{\vec{r}(t)\}$ is world line. In classical electromagnetism, we introduce potentials $\vec{A}, \phi$.

$$\begin{aligned} \vec{E} &= -\frac{\partial \vec{A}}{\partial t} - \nabla \phi \\ \vec{B} &= \nabla \times \vec{A} \end{aligned}$$

In component representation,

$$\begin{aligned} E_i &= -\frac{\partial \vec{A}}{\partial t} - \partial_i \phi, \\ B_i &= \varepsilon_{ijk} \partial_j A_k, \end{aligned}$$

where $i = x, y, z$.Rewriting motion of equation,

$$\begin{aligned} m\ddot{r}_i &= e\left(-\partial_0 A_i - \partial_i \phi + \varepsilon_{ijk} v_j \varepsilon_{klm} \partial_l A_m\right) \\ &= e\left(-\partial_0 A_i - \partial_i \phi + \varepsilon_{kij} v_j \varepsilon_{klm} \partial_l A_m\right) \\ &= e\left(-\partial_0 A_i - \partial_i \phi + \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right) v_j \partial_l A_m\right) \\ &= e\left(-\partial_0 A_i - \partial_i \phi + v_j \partial_i A_j - v_j \partial_j A_i\right). \end{aligned}$$

To smoothly quantization, let think principle of least action. Action is given by

$$S[\vec{r}] = \int_{t_1}^{t_2} dt L\left(\vec{r}(t), \dot{\vec{r}}(t)\right),$$

where L indicate Lagrangian. Following principle of least action,

$$\delta S = 0 \;\Leftrightarrow\; \text{Newton equation of motion.}$$

One suggestion of Lagrangian is

$$L = \frac{1}{2} m \dot{\vec{r}}^{2} - e\phi\left(r(t), t\right) + \dot{\vec{r}} \cdot \vec{A}\left(\vec{r}(t), t\right).$$

Let us check this Lagrangian. $\delta S = 0$ give us Euler-Lagrange equation,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}_i} - \frac{\partial L}{\partial r_i} = 0.$$

Substituting this Lagrangian,

$$\begin{aligned} \frac{\partial L}{\partial r_i} &= -e\partial_i \phi + e\dot{r}_j \cdot \partial_i A_j, \\ \frac{\partial L}{\partial \dot{r}_i} &= m\dot{r}_i + eA_i\left(\vec{r}(t), t\right), \\ \frac{d}{dt}\frac{\partial}{\partial \dot{r}_i} &= m\ddot{r}_i + e\left(\dot{r}_j \partial_j A_i + \partial_0 A_i\right). \end{aligned}$$

Then, Euler-Lagrange equation is given by

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{r}_i} - \frac{\partial L}{\partial r_i} &= m\ddot{r}_i + e\left(\dot{r}_j \partial_j A_i + \partial_0 A_i\right) - \left\{-e\partial_i \phi + e\dot{r}_j \cdot \partial_i A_j\right\} \\ &= 0, \end{aligned}$$

$$m\ddot{r}_i = e\left(-\partial_0 A_i - \partial_i \phi + e\dot{r}_j \cdot \partial_i A_j - \dot{r}_j \partial_j A_i\right).$$

This result is agreement with Newton equation.

A. <Gauge transformation>

There are ambiguity for choosing gauge. (i.e. $\phi, \vec{A}$ are not unique.)

$$\begin{aligned} \vec{B} &= \nabla \times \vec{A}, \\ \vec{E} &= -\partial_0 \vec{A} - \nabla \phi. \end{aligned}$$

If we take

$$\begin{aligned} \vec{A}' &:= \vec{A} + \nabla \chi, \\ \phi' &:= \phi - \partial_0 \chi. \end{aligned}$$

Here, χ is arbitrary single valued function. These $\vec{A}', \phi'$ give also same electromagnetic fields.

$$\because \vec{B}' = \nabla \times \vec{A}' = \nabla \times \vec{A} + \nabla \times (\nabla \chi) = \nabla \times \vec{A} = \vec{B}$$

$$\begin{aligned} \vec{E}' &= -\partial_0 \vec{A}' - \nabla \phi' \\ &= -\partial_0 \left(\vec{A} + \nabla \chi \right) - \nabla (\phi - \partial_0 \chi) \\ &= \vec{E} - (\partial_0 \nabla \chi - \nabla \partial_0 \chi) \\ &= \vec{E}. \end{aligned}$$

We used the fact,

$$(\nabla \times \nabla \chi)_i = \varepsilon_{ijk} \partial_j \partial_k \chi = 0,$$

and derivative for time and space are commute.

By gauge transformation, Lagrangian change as following

$$\begin{aligned} L' &= \frac{1}{2} m \dot{\vec{r}}^2 - e\phi' + e\dot{\vec{r}} \cdot \vec{A}' \\ &= \frac{1}{2} m \dot{\vec{r}}^2 - e(\phi - \partial_0 \chi) + e\dot{\vec{r}} \cdot \left(\vec{A} + \nabla \chi \right) \\ &= L + e\left(\partial_0 \chi + \dot{\vec{r}} \cdot \nabla \chi \right). \end{aligned}$$

Thus, Lagrangian not invariant under gauge transformation. However, this 2nd changed term can rewritten

$$e\left(\partial_0 \chi + \dot{\vec{r}} \cdot \nabla \chi \right) = e\frac{d}{dt}\chi.$$

Total time derivatives term can not affect the physics. Since, this term gives constant difference of action for fixed initial and final point,

$$\begin{aligned} S'[r] &= \int_{t_i}^{t_f} dt L' \\ &= \int_{t_i}^{t_f} dt L + \int_{t_i}^{t_f} dt e \frac{d}{dt} \chi \\ &= S[r] + e\left[\chi(t_f) - \chi(t_i) \right], \end{aligned}$$

and this constant cancel out when take variation.

To using canonical quantization, now we take canonical form by Legendre transformation. In the Hamiltonian mechanics,

$$\begin{aligned} H(\vec{r}, \vec{p}) &:= \dot{\vec{r}} \cdot \vec{p} - L\left(\vec{r}, \dot{\vec{r}}; t \right) \\ \vec{p}_i &:= \frac{\partial L}{\partial \dot{r}_i} \end{aligned}$$

(cf.) $H(\vec{r},\vec{p})$ is independent for $\dot{\vec{r}}$? ANS. yes!

$$\begin{aligned}\because\ \delta H &= \delta\dot{\vec{r}}\cdot\vec{p}+\dot{\vec{r}}\delta\vec{p}-\delta r\cdot\frac{\partial L}{\partial\vec{r}}-\delta\dot{\vec{r}}\frac{\partial L}{\partial\dot{\vec{r}}}\\ &= \delta\dot{\vec{r}}\cdot\left(\vec{p}-\frac{\partial L}{\partial\dot{\vec{r}}}\right)+\dot{\vec{r}}\delta\vec{p}-\delta r\cdot\frac{\partial L}{\partial\vec{r}}\\ &= \dot{\vec{r}}\delta\vec{p}-\delta r\cdot\frac{\partial L}{\partial\vec{r}}.\ \left(H\text{ is independent of }\dot{\vec{r}}\right)\end{aligned}$$

The Hamiltonian equation is given by

$$\begin{aligned}\frac{\partial H}{\partial p} &= \dot{\vec{r}}\\ \frac{\partial H}{\partial\vec{r}} &= -\dot{\vec{p}}.\end{aligned}$$

Because,

$$\begin{aligned}\frac{\partial H}{\partial\vec{p}} &= \dot{\vec{r}},\\ \frac{\partial H}{\partial\vec{r}} &= -\frac{\partial L}{\partial\vec{r}}=-\frac{d}{dt}\frac{\partial L}{\partial\dot{\vec{r}}}=-\dot{\vec{p}}.\end{aligned}$$

In electromagnetic system,

$$\begin{aligned}L\left(\vec{r},\dot{\vec{r}};t\right) &= \frac{1}{2}m\dot{\vec{r}}^{2}-e\phi+e\dot{\vec{r}}\cdot\vec{A}\\ \vec{p} &= \frac{\partial L}{\partial\dot{\vec{r}}}=m\dot{\vec{r}}+e\vec{A}\\ \dot{\vec{r}} &= \frac{1}{m}\left(\vec{p}-e\vec{A}\right).\end{aligned}$$

Under gauge transformation,

$$\begin{aligned}L' &= L+e\left(\partial_0\chi+\dot{\vec{r}}\nabla\chi\right)\\ \vec{p}' &= \frac{\partial L}{\partial\dot{\vec{r}}}+e\nabla\chi\\ &= \vec{p}+e\nabla\chi.\ \text{(Not invariant !)}\end{aligned}$$

This canonical momentum is not invariant under gauge transformation, but mechanical momentum is invariant :

$$\begin{aligned}\dot{\vec{r}}' &= \frac{1}{m}\left(\vec{p}'-e\vec{A}'\right)\\ &= \frac{1}{m}\left(\vec{p}+e\nabla\chi-e\left(\vec{A}+\nabla\chi\right)\right)\\ &= \dot{\vec{r}}+\frac{1}{m}\left(e\nabla\chi-e\nabla\chi\right)\\ &= \dot{\vec{r}}.\end{aligned}$$

The Hamiltonian is

$$\begin{aligned}H &= \dot{\vec{r}}\cdot\vec{p}-L\left(\vec{r},\dot{\vec{r}};t\right)\\ &= \dot{\vec{r}}\cdot\left(m\dot{\vec{r}}+e\vec{A}\right)-\left(\frac{1}{2}m\dot{\vec{r}}^{2}-e\phi+e\dot{\vec{r}}\cdot\vec{A}\right)\\ &= \frac{1}{2}m\dot{\vec{r}}^{2}+e\phi\\ &= \frac{1}{2m}\left(\vec{p}-e\vec{A}\right)^{2}+e\phi.\end{aligned}$$

II. QUANTUM THEORY - CANONICAL QUANTIZATION

$(\vec{r}, \vec{p})$: Canonical variables.
Canonical quantization is given by

$$[r_i, p_j] = \delta_{ij} i\hbar,$$

where, $[A, B] := AB - BA$, $\hbar$ indicate Plank constant.
In positional space, classical variable change as following operators

$$\begin{aligned} x &\rightarrow \hat{x} \\ p &\rightarrow \hat{p} = -\hbar \frac{\partial}{\partial x}. \end{aligned}$$

Then, we get the Schrodinger equation as followings

$$i\hbar \frac{\partial}{\partial t}\psi = \frac{1}{2m}\left(-i\hbar\nabla - e\vec{A}\right)^2 \psi + e\phi\psi.$$

Part III
Electron in a magnetic field (2007/09/18)

Our target is explain QHE(Quantum Hall effects). As we discussed last class, it related Hall conductance. When one put magnetic field $\vec{B} = B\hat{z}$ on 2 dimensionally confined electron system. If there exist current $I = I\hat{x}$, there are voltage to contain current, since charge carrier feels Lorentz force. These relation represents as following :

$$I_x = \sigma_{xy} V_y,$$

where σ_{xy} indicates Hall conductance. Classically,

$$\sigma_{xy} = n\frac{e}{h},$$

where n is an integer. But if there are Quantum Hall effects, n change to p/q(fractional number).

III. REVIEW OF CLASSICAL DESCRIPTION

As we saw in the last class, Lagrangian is given by

$$L = \frac{1}{2}m\dot{\vec{r}}^2 - e\phi + e\dot{\vec{r}} \cdot \vec{A}.$$

And canonical momentum is given by

$$\begin{aligned} \vec{p} \ &: = \frac{\partial L}{\partial \dot{\vec{r}}} = m\dot{\vec{r}} + e\vec{A} \\ &= m\vec{v} + e\vec{A}, \end{aligned}$$

where $\vec{v} := \dot{\vec{r}}$.

$$\dot{\vec{r}} = \frac{1}{m}\left(\vec{p} - e\vec{A}\right).$$

Hamiltonian is

$$\begin{aligned} H &= \vec{p} \cdot \dot{\vec{r}} - L \\ &= \left(m\dot{\vec{r}}^2 + e\vec{A} \cdot \dot{\vec{r}}\right) - \left(\frac{1}{2}m\dot{\vec{r}}^2 - e\phi + e\dot{\vec{r}} \cdot \vec{A}\right) \\ &= \frac{1}{2}m\dot{\vec{r}}^2 + e\phi \\ &= \frac{1}{2m}\left(\vec{p} - e\vec{A}\right)^2 + e\phi. \end{aligned}$$

(comment)

$$\begin{aligned}\vec{j} &= e\dot{\vec{r}} \\ &= \frac{\partial L}{\partial \vec{A}} \\ &= -\frac{\partial H}{\partial \vec{A}} = -\frac{1}{m}\left(\vec{p}-e\vec{A}\right)(-e)\,.\end{aligned}$$

IV. REVIEW OF QUANTUM DESCRIPTION

Using canonical quantization,

$$[r_\alpha, p_\beta] = i\hbar\delta_{\alpha\beta},$$

and in position space, momentum operator is given by

$$\vec{p} = \frac{\hbar}{i}\nabla,$$

we can write Schrodinger equation

$$\begin{aligned}i\hbar\frac{\partial\psi}{\partial t} &= H\left(\vec{r},\vec{p}\right)\psi \\ &= H\left(\vec{r},\frac{\hbar}{i}\nabla\right)\psi,\end{aligned}$$

where $\psi\left(\vec{r},t\right)$ is a wave function.

V. GAUGE TRANSFORMATION & CONSERVED CURRENT(SPIN HALL EFFECT)

(Comment) Laughlim argument : essence of the QHE.
As well known Nother theory, symmetry gives conserved property.
We will see that the probability of density is conserved as in the level of quantum mechanics.
First, similarly to classical mechanics, we define current density as

$$\begin{aligned}\vec{j}' &:= \psi^* e\vec{v}\psi \\ &= \frac{e}{m}\psi^*\left(\vec{p}-e\vec{A}\right)\psi \\ &= \frac{e}{m}\psi^*\left(\frac{\hbar}{i}\nabla - e\vec{A}\right)\psi.\end{aligned}$$

Integrate for all space,

$$\begin{aligned}\int d\vec{r}\,\vec{j}' &= \int d\vec{r}\frac{e}{m}\psi^*\left(\frac{\hbar}{i}\nabla - e\vec{A}\right)\psi \\ &= \int d\vec{r}\frac{e}{m}\psi^*\frac{\hbar}{i}\nabla\psi - \int d\vec{r}\frac{e^2}{m}\vec{A}\left|\psi\right|^2 \\ &= \int d\vec{r}\frac{e\hbar}{mi}\left[\nabla\left(\psi^*\psi\right) - \left(\nabla\psi^*\right)\psi\right] - \int d\vec{r}\frac{e^2}{m}\vec{A}\left|\psi\right|^2 \\ &= \left[\frac{e\hbar}{mi}\left(\psi^*\psi\right)\right]_{\text{surface}} - \int d\vec{r}\frac{e\hbar}{mi}\left(\nabla\psi^*\right)\psi - \int d\vec{r}\frac{e^2}{m}\vec{A}\left|\psi\right|^2 \\ &= -\int d\vec{r}\frac{e\hbar}{mi}\left(\nabla\psi^*\right)\psi - \int d\vec{r}\frac{e^2}{m}\vec{A}\left|\psi\right|^2\,.\end{aligned}$$

Also we can take

$$\vec{j}'' := \psi e\vec{v}\psi^*.$$

And, to make well defined current we now define current density as following :

$$\begin{aligned}\vec{j} \; : \; &= \frac{1}{2}\left(\vec{j}' + \vec{j}''\right) \\ &= \frac{\hbar e}{2mi}\left(\left(\nabla\psi\right)\psi^* - \left(\nabla\psi^*\right)\psi\right) - \frac{e^2\vec{A}}{m}\left|\psi\right|^2 . \end{aligned} \tag{1}$$

Charge density is given by

$$\rho\left(\vec{r}\right) = e\left|\psi\left(\vec{r}\right)\right|^2 = e\psi^*\psi,$$

where $\left|\psi\left(\vec{r}\right)\right|^2$ is a probability density of the particle at $\vec{r}$. Differentiating $\rho\left(\vec{r}\right)$ by time t, we get

$$\frac{\partial\rho}{\partial t} = e\left(\frac{\partial\psi^*}{\partial t}\psi + \psi^*\frac{\partial\psi}{\partial t}\right). \tag{2}$$

Using the Schrodinger equation,

$$\begin{aligned}\frac{\partial\psi}{\partial t} &= \frac{1}{i\hbar}\left[\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - e\vec{A}\right)^2\psi + e\phi\psi\right] \\ &= \frac{1}{i\hbar}\left[\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - e\vec{A}\right)\left(\frac{\hbar}{i}\nabla - e\vec{A}\right)\psi + e\phi\psi\right] \\ &= \frac{1}{i\hbar}\left[\frac{1}{2m}\left(\left(\frac{\hbar}{i}\nabla\right)^2 - \frac{\hbar}{i}\left\{\nabla\cdot\left(e\vec{A}\right)\right\} - \frac{2\hbar}{i}e\vec{A}\nabla + e^2\vec{A}^2\right)\psi + e\phi\psi\right],\end{aligned}$$

and its complex conjugate

$$\begin{aligned}\frac{\partial\psi^*}{\partial t} &= \frac{-1}{i\hbar}\left[\frac{1}{2m}\left(-\frac{\hbar}{i}\nabla - e\vec{A}\right)^2\psi^* + e\phi\psi^*\right] \\ &= \frac{-1}{i\hbar}\left[\frac{1}{2m}\left(\left(\frac{\hbar}{i}\nabla\right)^2 + \frac{\hbar}{i}\left\{\nabla\cdot\left(e\vec{A}\right)\right\} + \frac{2\hbar}{i}e\vec{A}\nabla + e^2\vec{A}^2\right)\psi^* + e\phi\psi^*\right],\end{aligned}$$

we can rewrite (Eq. 2) as

$$\begin{aligned}\frac{\partial\rho}{\partial t} &= \frac{e}{i\hbar}\psi^*\left[\frac{1}{2m}\left(\left(\frac{\hbar}{i}\nabla\right)^2 - \frac{\hbar}{i}\left\{\nabla\cdot\left(e\vec{A}\right)\right\} - \frac{2\hbar}{i}e\vec{A}\nabla + e^2\vec{A}^2\right)\psi + e\phi\psi\right] \\ &\quad -\frac{e}{i\hbar}\psi\left[\frac{1}{2m}\left(\left(\frac{\hbar}{i}\nabla\right)^2 + \frac{\hbar}{i}\left\{\nabla\cdot\left(e\vec{A}\right)\right\} + \frac{2\hbar}{i}e\vec{A}\nabla + e^2\vec{A}^2\right)\psi^* + e\phi\psi^*\right] \\ &= \frac{e}{i\hbar}\psi^*\left[\frac{1}{2m}\left(\left(\frac{\hbar}{i}\nabla\right)^2 - \frac{\hbar}{i}\left\{\nabla\cdot\left(e\vec{A}\right)\right\} - \frac{2\hbar}{i}e\vec{A}\nabla\right)\psi\right] \\ &\quad -\frac{e}{i\hbar}\psi\left[\frac{1}{2m}\left(\left(\frac{\hbar}{i}\nabla\right)^2 + \frac{\hbar}{i}\left\{\nabla\cdot\left(e\vec{A}\right)\right\} + \frac{2\hbar}{i}e\vec{A}\nabla\right)\psi^*\right] \\ &= \frac{e}{i\hbar}\frac{1}{2m}\left[\psi^*\left(\frac{\hbar}{i}\nabla\right)^2\psi - \psi\left(\frac{\hbar}{i}\nabla\right)^2\psi^* - \frac{2\hbar}{i}e\vec{A}\left(\psi^*\nabla\psi + \psi\nabla\psi^*\right) - \frac{2\hbar}{i}\left\{\nabla\cdot\left(e\vec{A}\right)\right\}\left|\psi\right|^2\right] \\ &= \frac{-e}{i\hbar}\frac{1}{2m}\left[\hbar^2\left(\psi^*\left(\nabla\right)^2\psi - \psi\left(\nabla\right)^2\psi^*\right) + \frac{2\hbar}{i}e\vec{A}\left(\psi^*\nabla\psi + \psi\nabla\psi^*\right) + \frac{2\hbar}{i}\left\{\nabla\cdot\left(e\vec{A}\right)\right\}\left|\psi\right|^2\right] \\ &= \frac{-e}{i\hbar}\frac{1}{2m}\left[\hbar^2\nabla\cdot\left(\psi^*\left(\nabla\psi\right) - \psi\left(\nabla\psi^*\right)\right) + \frac{2\hbar}{i}e\vec{A}\nabla\cdot\left(\psi^*\psi\right) + \frac{2\hbar}{i}\left\{\nabla\cdot\left(e\vec{A}\right)\right\}\left|\psi\right|^2\right] \\ &= \frac{-e}{i\hbar}\frac{1}{2m}\left[\hbar^2\nabla\cdot\left(\psi^*\left(\nabla\psi\right) - \psi\left(\nabla\psi^*\right)\right) + \frac{2\hbar}{i}e\vec{A}\nabla\cdot\left(\psi^*\psi\right) + \frac{2\hbar}{i}\left\{\nabla\cdot\left(e\vec{A}\right)\right\}\left|\psi\right|^2\right] \\ &= \frac{-e}{i\hbar}\frac{1}{2m}\left[\hbar^2\nabla\cdot\left(\psi^*\left(\nabla\psi\right) - \psi\left(\nabla\psi^*\right)\right) + \frac{2\hbar}{i}e\nabla\cdot\left(\vec{A}\psi^*\psi\right)\right] \\ &= -\nabla\cdot\left[\frac{\hbar e}{2mi}\left(\psi^*\left(\nabla\psi\right) - \psi\left(\nabla\psi^*\right)\right) - \frac{e}{m}\left(\vec{A}\psi^*\psi\right)\right].\end{aligned}$$

From the definition of current density (Eq. 1),

$$\vec{j} = \frac{\hbar e}{2mi}\left(\left(\nabla\psi\right)\psi^* - \left(\nabla\psi^*\right)\psi\right) - \frac{e^2\vec{A}}{m}\left|\psi\right|^2,$$

we finally get equation of continuity,

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot\vec{j}.$$

It represents conservation of probability.

Part IV
(2007/09/25)

VI. REVIEW

$$i\hbar\frac{\partial}{\partial t}\psi = \psi H,$$

where,

$$\begin{aligned} H &= \frac{1}{2m}\left(\vec{p} - e\vec{A}\right)^2 + e\phi, \\ \vec{p} &= \frac{\hbar}{i}\nabla. \end{aligned}$$

Density $\rho = e\left|\psi\right|^2$
Density current $\vec{j} = e\left[\frac{\hbar}{2m}\left(\psi^*\nabla\psi - \left(\nabla\psi^*\right)\psi\right) - \frac{e}{m}\vec{A}\left|\psi\right|^2\right]$
Conservation law $\partial_0\rho + \nabla\cdot\vec{j} = 0$.

VII. GAUGE TRANSFORMATION

Let us see gauge transformation in quantum mechanical system.
Gauge transformation

$$\begin{aligned} \vec{A} &\rightarrow \vec{A}' = \vec{A} + \nabla\chi \\ \phi &\rightarrow \phi' = \phi - \frac{\partial\chi}{\partial t} \end{aligned}$$

where χ is a single valued function. In gauge transformation, Hamiltonian and wave function will be changed.
Assume local gauge transformation form,

$$\psi'\left(\vec{r},t\right) = e^{i\theta(\vec{r},t)}\psi\left(\vec{r},t\right),$$

this condition represents the phase is arbitrary at each point of space-time. (Gauge symmetry) Then,

$$\psi\left(\vec{r},t\right) = e^{-i\theta(\vec{r},t)}\psi'\left(\vec{r},t\right),$$

and we see that

$$\begin{aligned} \left(\vec{p} - e\vec{A}\right)\psi &= \left(-i\hbar\nabla - e\vec{A}\right)e^{-i\theta}\psi' \\ &= \left(-i\hbar\right)\left(-i\nabla\theta\right)e^{-i\theta}\psi' + \left(-i\hbar\right)e^{-i\theta}\nabla\psi' - e\vec{A}\,e^{-i\theta}\psi' \\ &= e^{-i\theta}\left[-\hbar\nabla\theta - i\hbar\nabla - e\vec{A}\right]\psi' \\ &= e^{-i\theta}\left[-i\hbar\nabla - e\left(\vec{A} + \frac{\hbar}{e}\nabla\theta\right)\right]\psi' \\ &= e^{-i\theta}\left[\vec{p} - e\vec{A}'\right]\psi', \end{aligned}$$

last line I defined $\vec{A}' := \vec{A} + \frac{\hbar}{e}\nabla\theta, (\theta := \frac{e}{\hbar}\chi)$.

Since

$$\begin{aligned}\left(\vec{p} - e\vec{A}\right)^2 \psi &= \left(\vec{p} - e\vec{A}\right)\left(\vec{p} - e\vec{A}\right)\psi \\ &= \left(\vec{p} - e\vec{A}\right)\left\{e^{-i\theta}\left[\vec{p} - e\vec{A}'\right]\psi'\right\} \\ &= e^{-i\theta}\left(\vec{p} - e\vec{A}'\right)^2 \psi'.\end{aligned}$$

Recalling $\psi' = e^{i\theta}\psi$,

$$\begin{aligned}i\hbar\frac{\partial}{\partial t}\psi &= i\hbar\frac{\partial}{\partial t}\left(e^{-i\theta}\psi'\right) \\ &= i\hbar(-i)\left(\frac{\partial\theta}{\partial t}\right)\psi' + i\hbar e^{-i\theta}\frac{\partial}{\partial t}\psi' \\ &= \hbar\frac{\partial\theta}{\partial t}\psi' + i\hbar e^{-i\theta}\frac{\partial}{\partial t}\psi'.\end{aligned}$$

Then, the Schrodinger equation,

$$i\hbar\frac{\partial}{\partial t}\psi = \left[\frac{1}{2m}\left(\vec{p} - e\vec{A}\right)^2 + e\phi\right]\psi,$$

$\psi = e^{-i\theta}\psi'$,

$$\hbar\frac{\partial\theta}{\partial t}\psi' e^{-i\theta} + i\hbar e^{-i\theta}\frac{\partial}{\partial t}\psi' = \frac{e^{-i\theta}}{2m}\left(\vec{p} - e\vec{A}'\right)^2\psi' + e\phi e^{-i\theta}\psi',$$

multiplying $e^{i\theta}$,

$$\begin{aligned}i\hbar\frac{\partial\psi'}{\partial t} &= \left[\frac{1}{2m}\left(\vec{p} - e\vec{A}'\right)^2\psi' + e\phi\psi' - \hbar\frac{\partial\theta}{\partial t}\psi'\right] \\ &= \left[\frac{1}{2m}\left(\vec{p} - e\vec{A}'\right)^2 + e\left(\phi - \frac{\hbar}{e}\frac{\partial\theta}{\partial t}\right)\right]\psi' \\ &= \left[\frac{1}{2m}\left(\vec{p} - e\vec{A}'\right)^2 + e\left(\phi - \frac{\partial\chi}{\partial t}\right)\right]\psi' \\ &= \left[\frac{1}{2m}\left(\vec{p} - e\vec{A}'\right)^2 + e\phi'\right]\psi' \\ &= H'\psi'.\end{aligned}$$

Summarizing these,

$$\begin{aligned}\vec{A} &\to \vec{A}' = \vec{A} + \nabla\chi \\ \phi &\to \phi' = \phi - \frac{\partial\chi}{\partial t} \\ \psi &\to \psi' = e^{i\theta}\psi \\ H &\to H' = \frac{1}{2m}\left(\vec{p} - e\vec{A}'\right)^2 + e\phi' \\ i\hbar\frac{\partial\psi}{\partial t} &= H\psi \\ i\hbar\frac{\partial\psi'}{\partial t} &= H'\psi'\end{aligned}$$

we see that Schrodinger equation is covariant. Observables like density of probability or density current need to invariant. Let us confirm these, density of probability is clearly,

$$\rho \to \rho' = \left|\psi'\right|^2 = |\psi|^2.$$

$$\vec{j} \to \vec{j}' = e\left[\frac{\hbar}{2mi}\left(\psi'^{*}\nabla\psi' - \left(\nabla\psi'^{*}\right)\psi'\right) - \frac{e}{m}\vec{A}'\left|\psi'\right|^2\right]$$

$$\begin{aligned}\vec{j}' &= e\left[\frac{\hbar}{2mi}\left(\psi^{*}\nabla\psi - \left(\nabla\psi^{*}\right)\psi + 2i\nabla\theta\left|\psi\right|^2\right) - \frac{e}{m}\left(\vec{A} + \nabla\chi\right)\left|\psi\right|^2\right] \\ &= \vec{j} + e\rho\left(\frac{\hbar}{2mi}2i\nabla\theta - \frac{e}{m}\nabla\chi\right) \\ &= \vec{j}.\end{aligned}$$

Thus, conservation law is conserved

$$\frac{\partial\rho'}{\partial t} + \nabla'\cdot\vec{j} = 0.$$

$$\begin{aligned}\psi &\to \psi' = \psi e^{i\frac{e}{\hbar}\chi} \\ &= \psi e^{i2\pi\frac{e}{h}\chi} \\ &: = \psi e^{i2\pi\frac{\chi}{\Phi_0}},\end{aligned}$$

where $\Phi_0 := \frac{h}{e}$ is called flux quantum.

Dimensional analysis of $\vec{A}$: From

$$\vec{A} \to \vec{A}' = \vec{A} + \nabla\chi,$$

$$\vec{B} = \nabla\times\vec{A}, .$$

we see magnetic field and χ has following dimensional relation:

$$[B] \sim \frac{[\chi]}{[L^2]},$$

$$\begin{aligned}\chi &\sim B\left[L^2\right] \sim \text{flux} \\ \chi &\equiv \chi' \ [\text{mod } \Phi_0].\end{aligned}$$

<Example> Let us assume that there exist uniform magnetic field $\vec{B} = \nabla\times\vec{A} = (0,0,B)$(time independent) with $\vec{E} = 0$.

Gauge choice 1.Landau gauge $\phi = 0, \vec{A}_L = (0, xB, 0)$

Gauge choice 2.Symmetric gauge $\phi = 0, \vec{A}_S = \frac{1}{2}(yB, -xB, 0)$

Both gauges give us

$$\vec{B} = \nabla\times\vec{A} = (0,0,B).$$

In steady state, we can separate variables,

$$\psi(r,t) = e^{-i\omega t}\psi(\vec{r}),$$

and we have

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi,$$

defining $E := \hbar\omega$, we see that

$$\hbar\omega = E.$$

Using Landau gauge,

$$\frac{1}{2m}\left(\vec{p} - e\vec{A}_L\right)^2\psi = \frac{1}{2m}\left[-\hbar^2\frac{\partial^2}{\partial x^2} + \left(-i\hbar\frac{\partial}{\partial y} + eB_x\right)^2 - \hbar^2\frac{\partial}{\partial z^2}\right]\psi = E\psi.$$

Inserting

$$\psi = e^{ik_y y} e^{ik_z z} \Phi(x),$$

we have

$$\frac{1}{2m}\left(\vec{p} - e\vec{A}_L\right)^2 \psi = \frac{1}{2m}\left[-\hbar^2 \Phi'' + (\hbar k_y + eB_x)^2 \Phi + \hbar^2 k_z^2 \Phi\right] e^{ik_y y} e^{ik_z z}.$$

Defining

$$E_{2D} = E - \frac{\hbar^2 k_z^2}{2m},$$

we can compare harmonics oscillator

$$\frac{\hbar^2}{2m}\left[-\frac{d^2}{dx^2} + \left(k_y + \frac{e}{\hbar} Bx\right)^2\right] \Phi = E_{2D}\Phi, \tag{3}$$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2 \text{: cf. Harmonic oscillator.}$$

Putting

$$\begin{aligned} X &= x + \frac{\hbar}{eB} k_y, \\ \frac{d}{dX} &= \frac{d}{dx}, \end{aligned}$$

(Eq. 3) is given by

$$\begin{aligned} \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{\hbar^2}{2m}\frac{e^2 B^2}{\hbar^2}\left(x + \frac{\hbar}{eB}k_y\right)^2\right]\Phi &= E_{2D}\Phi, \\ \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{m}{2}\frac{e^2 B^2}{m^2} X^2\right]\Phi &= E_{2D}\Phi. \end{aligned}$$

So,

$$\omega = \left(\frac{eB}{m}\right)$$

is called as cycrotron frequency. And E_{2D} is quantize, it called Landau levels,

$$\begin{aligned} E_{2D} &= \frac{1}{2}\hbar\omega\left(n + \frac{1}{2}\right), \\ n &= 0, 1, 2, ... \end{aligned}$$

$$X = x + \frac{\hbar k_y}{eB} =: x + l\ k_y l,$$

note that $k_y l$ is dimensionless.

$$\begin{aligned} l^2 &= \frac{\hbar}{eB}, \\ l &= \sqrt{\frac{\hbar}{eB}} \text{ : magnetic length.} \end{aligned}$$

note that magnetic length has a unique length scale, $\hbar\omega$ energy scale.

If there exists $O \neq 0$ Hermit operator which satisfied

$$[H, O] = 0,$$

then there exist conserved quantity. (Landau degeneracy)

Part V
(2007/10/02)

VIII. 2 DIMENSIONAL ELECTRON PROBLEM IN A UNIFORM MAGNETIC FIELD

Hamiltonian is given by

$$\begin{aligned} H &= \frac{1}{2m}\left(\vec{p} - e\vec{A}\right)^2 = \frac{1}{2m}\left[(p_x - eA_x)^2 + (p_y - eA_y)^2\right] \\ &= \frac{1}{2m}\vec{\Pi}^2, \end{aligned}$$

where, $\vec{\Pi} = (\Pi_x, \Pi_y), \Pi_i = (p_i - eA_i)$. We will think uniform magnetic field,

$$\begin{aligned} \vec{B} &= \nabla \times \vec{A} \\ &= (0, 0, B), \\ \left(\nabla \times \vec{A}\right)_z &= \partial_x A_y - \partial_y A_x = B. \end{aligned}$$

There are some commutation relations.

$$\begin{aligned} [\Pi_x, \Pi_y] &= [p_x - eA_x, p_y - eA_y] \\ &= [p_x, -eA_y] - [eA_x, p_y] \\ &= -e\{[p_x, A_y] + [A_x, p_y]\} \\ &= -e\{-i\hbar\partial_x A_y - i\hbar\partial_y A_x\} \\ &= i\hbar eB. \end{aligned}$$

$$\therefore [\Pi_x, \Pi_y] = i\hbar eB. \tag{4}$$

The last line I used

$$[p_i, G(\vec{x})] = -i\hbar\frac{\partial G}{\partial x_i}.$$

Let

$$\begin{aligned} l &\equiv \sqrt{\frac{\hbar}{eB}}, \\ l^2 &= \frac{\hbar}{eB},\ eB = \frac{\hbar}{l^2} \end{aligned}$$

then, (Eq.4) is rewritten as

$$\left[\frac{l}{\hbar}\Pi_x, \frac{l}{\hbar}\Pi_y\right] = i. \tag{5}$$

Defining

$$a \equiv \frac{1}{\sqrt{2}}\frac{l}{\hbar}(\Pi_x + i\Pi_y),$$

then, thinking $\Pi_i = \Pi_i^\dagger$, we get

$$a^\dagger = \frac{1}{\sqrt{2}}\frac{l}{\hbar}\left(\Pi_x - i\Pi_y\right),$$

and

$$\begin{aligned}\Pi_x &= \frac{1}{\sqrt{2}}\frac{\hbar}{l}\left(a + a^\dagger\right),\\ \Pi_y &= \frac{1}{i\sqrt{2}}\frac{\hbar}{l}\left(a - a^\dagger\right).\end{aligned}$$

(Eq.5) is written as

$$\frac{1}{2i}\left[\left(a + a^\dagger\right), \left(a - a^\dagger\right)\right] = i$$

$$\begin{aligned}(\text{L.H.S}) &= \frac{1}{2i}\left\{\left[a, -a^\dagger\right] + \left[a^\dagger, a\right]\right\}\\ &= \frac{1}{i}\left[a^\dagger, a\right] = i\left[a, a^\dagger\right]\end{aligned}$$

$$\therefore \left[a, a^\dagger\right] = 1.$$

It represents $a, a^\dagger$ operators are bosonic operators. Using these operators, Hamiltonian is

$$\begin{aligned}H &= \frac{1}{2m}\vec{\Pi}^2 = \frac{1}{2m}\left(\Pi_x^2 + \Pi_y^2\right)\\ &= \frac{1}{2m}\left\{\frac{1}{2}\frac{\hbar^2}{l^2}\left(a + a^\dagger\right)^2 - \frac{1}{2}\frac{\hbar^2}{l^2}\left(a - a^\dagger\right)^2\right\}\\ &= \frac{1}{4m}\frac{\hbar^2}{l^2}\left\{aa^\dagger + a^\dagger a\right\} \times 2\\ &= \frac{1}{m}\frac{\hbar^2}{l^2}\left\{a^\dagger a + \frac{1}{2}\right\}.\end{aligned}$$

At the last line, I used $\left[a, a^\dagger\right] = 1$. From the definition of $l = \sqrt{\frac{\hbar}{eB}}$,

$$\begin{aligned}H &= \frac{1}{m}\frac{\hbar^2}{l^2}\left\{a^\dagger a + \frac{1}{2}\right\}\\ &= \frac{\hbar^2}{m}\frac{eB}{\hbar}\left\{a^\dagger a + \frac{1}{2}\right\}\\ &= \hbar\omega\left(a^\dagger a + \frac{1}{2}\right),\end{aligned}$$

where $\omega \equiv \frac{eB}{m}$. It is Harmonic oscillator's Hamiltonian.

$$\begin{aligned}H &= \hbar\omega\left(\hat{n} + \frac{1}{2}\right),\\ \hat{n} &= a^\dagger a : \text{number operator}\end{aligned}$$

Eigenvalue $n = 0, 1, 2, 3, \cdots, E_n = \hbar\omega\left(n + \frac{1}{2}\right)$.
Eigenstate

$$\begin{aligned}|n\rangle &= \frac{1}{\sqrt{n!}}\left(a^\dagger\right)^n|0\rangle,\\ a|0\rangle &= 0.\end{aligned}$$

A. Guiding center operators

$$R_x \ : \ = x + \frac{l^2}{\hbar}\Pi_y,$$
$$R_y \ : \ = y - \frac{l^2}{\hbar}\Pi_x$$

(cf.) Dimension analysis

$$\Pi \sim p \sim \hbar k,$$
$$[\Pi] \propto [\hbar]\left[\frac{1}{L}\right]$$
$$\frac{l^2}{\hbar}\Pi \sim \frac{l^2}{\hbar}[\hbar]\left[\frac{1}{L}\right] \sim [L]$$

There are some commutation relations

$$[x, \Pi_x] = [x, (p_x - eA_x)] = [x, p_x] = i\hbar.$$

$$[y, \Pi_y] = i\hbar.$$

$$[x, \Pi_y] = [y, \Pi_x] = 0.$$

$$[\Pi_x, \Pi_y] = i\hbar eB$$

We can easily see that

$$\left[H, \vec{R}\right] = 0.$$

To see this, first we consider

$$\begin{aligned}
[H, R_x] &= \left[\frac{1}{2m}\left(\Pi_x^2 + \Pi_y^2\right), x + \frac{l^2}{\hbar}\Pi_y\right] \\
&= \frac{1}{2m}\left\{\left[\Pi_x^2, x\right] + \frac{l^2}{\hbar}\left[\Pi_x^2, \Pi_y\right]\right\} \\
&= \frac{1}{2m}\left\{\Pi_x\left[\Pi_x, x\right] - \left[x, \Pi_x\right]\Pi_x + \frac{l^2}{\hbar}\left(\Pi_x\left[\Pi_x, \Pi_y\right] - \left[\Pi_y, \Pi_x\right]\Pi_x\right)\right\} \\
&= \frac{1}{2m}\left\{\Pi_x(-i\hbar) - (i\hbar)\Pi_x + \frac{l^2}{\hbar}\left(\Pi_x i\hbar eB\right) \times 2\right\} \\
&= \frac{1}{2m}\left\{\Pi_x(-2i\hbar) + \frac{l^2}{\hbar}\left(\Pi_x i\hbar\frac{\hbar}{l^2}\right) \times 2\right\} = 0.
\end{aligned}$$

Similarly

$$[H, R_y] = 0.$$

Thus we see

$$\left[H, \vec{R}\right] = 0.$$

This means H and $\vec{R}$ operator has simultaneous ket.

(cf1) Review of quantum mechanics.
In general,

$$O: \text{ Hermitian operator. } \left(O = O^\dagger\right)$$

Then

$$\begin{aligned} O|\alpha\rangle &= \alpha|\alpha\rangle \quad \alpha\text{:real} \\ \langle\alpha|O^\dagger &= \langle\alpha|O = \alpha\langle\alpha|. \end{aligned}$$

Assume

$$[H, O] = 0.$$

For $\alpha \neq \beta$,

$$\langle\alpha|\,[H, O]\,|\beta\rangle = 0,$$

$$\begin{aligned} \langle\alpha|\,[H, O]\,|\beta\rangle &= \langle\alpha|\,(HO - OH)\,|\beta\rangle \\ &= (\alpha - \beta)\,\langle\alpha|H|\beta\rangle = 0. \end{aligned}$$

$$\therefore \langle\alpha|H|\beta\rangle = 0.$$

This means eigenkets of operator O are also eigenkets of H, since these eigenkets diagonalize H.

$$\begin{aligned} &\exists|n, \alpha\rangle \text{ s.t.} \\ H|n, \alpha\rangle &= E_n|n, \alpha\rangle \\ O|n, \alpha\rangle &= \alpha|n, \alpha\rangle. \end{aligned}$$

(cf2.) In our system,

$$\begin{aligned} \left[H, \vec{R}\right] &= 0, \\ [H, R_x] &= 0, \\ [H, R_y] &= 0. \end{aligned}$$

Here one should note that

$$[R_x, R_y] \neq 0$$

$$\begin{aligned} [R_x, R_y] &= \left[x + \frac{l^2}{\hbar}\Pi_y, y - \frac{l^2}{\hbar}\Pi_x\right] \\ &= -\frac{l^2}{\hbar}\,[x, \Pi_x] + \frac{l^2}{\hbar}\,[\Pi_y, y] - \frac{l^4}{\hbar^2}\,[\Pi_y, \Pi_x] \\ &= -\frac{l^2}{\hbar}\,(i\hbar) + \frac{l^2}{\hbar}\,(-i\hbar) - \frac{l^4}{\hbar^2}\,(-i\hbar eB) \\ &= -\frac{l^4}{\hbar^2}\left(-i\hbar\frac{\hbar}{l^2}\right) \\ &= il^2. \end{aligned}$$

$$\therefore \left[\frac{R_x}{l}, \frac{R_y}{l}\right] = i.$$

(cf3)

$$\left[\frac{l}{\hbar}\Pi_x, \frac{l}{\hbar}\Pi_y\right] = i$$

$$a := \frac{1}{\sqrt{2}}\left(\frac{l}{\hbar}\Pi_x + i\frac{l}{\hbar}\Pi_y\right), \qquad \left[a, a^\dagger\right] = 0$$
$$b := \frac{1}{\sqrt{2}}\left(\frac{R_y}{l} + i\frac{R_x}{l}\right), \qquad \left[b, b^\dagger\right] = 0.$$

$$\begin{aligned} H|n,m\rangle &= E_n|n,m\rangle \\ |n,m\rangle &= \frac{1}{\sqrt{n!}}\frac{1}{\sqrt{m!}}\left(a^\dagger\right)^n (a)^m |0\rangle, \\ E_n &= \hbar\omega\left(n + \frac{1}{2}\right) \quad : m \text{ independent} \\ H &= \hbar\omega\left(a^\dagger a + \frac{1}{2}\right), \ m = 0,1,2,\cdots \quad \Rightarrow \text{Landau degeneracy} \end{aligned}$$

IX. 2 DIMENSIONAL BLOACH ELECTRONS IN AN UNIFORM MAGNETIC FIELD(PREVIEW OF NEXT CLASS)

Thinking 2 dimensional lattice with lattice constant a. At $a \to 0$, we get continuum limit. Gauge transformation,

$$\begin{aligned} \vec{A} &\to \vec{A}' = \vec{A} + \nabla\chi, \\ \psi &\to \psi' = e^{i2\pi\frac{1}{\Phi_0}\chi}\psi. \end{aligned}$$

If one think gauge

$$\vec{A}' = 0 \ \text{(local)}$$

then,

$$\vec{A} + \nabla\chi = 0,$$

$$\begin{aligned} \chi &= -\int_{r_0}^{r} d\vec{r}\cdot\vec{A}, \\ \psi' &= e^{-i2\pi\frac{1}{\Phi_0}\int_{r_0}^{r} d\vec{r}\cdot\vec{A}}\psi. \end{aligned}$$

Hamiltonian

$$H = T_x + T_y + \text{h.c.}$$

$$T_x = \sum_{m,n} te^{i\theta^x_{mn}} C^\dagger_{m+1\ n} C_{m\ n},$$

where m, n represent lattice point.

$$\begin{aligned} T_y &= \sum_{m,n} te^{i\theta^y_{mn}} C^\dagger_{m\ n+1} C_{mn}, \\ \left\{C^\dagger_{mn}, C_{m'n'}\right\} &= \delta_{mm'}\delta_{nn'}. \end{aligned}$$

One particle state

$$\begin{aligned} |\psi\rangle &= \sum_{m\ n} \psi_{mn} C^\dagger_{mn}|0\rangle \\ H|\psi\rangle &= E|\psi\rangle. \end{aligned}$$

$\{\psi_{mn}\}$ has linear equation

$$\begin{aligned} \psi_{nm} &= \psi\left(\vec{r}_{mn}\right), \\ \vec{r}_{mn} &= m\,(a,0) + n\,(0,a) \end{aligned}$$

$\psi(\vec{r})$ satisfy

$$\frac{1}{2m}\vec{\Pi}^2\psi(r) = E\psi(r)$$

in continuum limit.

Part VI
(2007/10/16)

X. QUANTUM HALL EFFECTS

Square lattice, lattice point (m, n).
x-directional Translation operator

$$T_x = \sum_{m,n} te^{i\theta^x_{mn}} C^\dagger_{m=1,n} C_{m,n}.$$

y-directional Translation operator

$$T_y = \sum_{m,n} te^{i\theta^y_{mn}} C^\dagger_{m+1,n} C_{m,n}.$$

One particle state

$$|\psi\rangle = \sum_{m,n} \psi_{m\ n} C_{m\ n}|0\rangle.$$

Schrodinger equation

$$H|\psi\rangle = E|\psi\rangle.$$

$$\begin{aligned} T^\dagger_x &= \left(\sum te^{i\theta^x_{mn}} C^\dagger_{m+1\ n} C_{m\ n}\right)^\dagger \\ &= \sum te^{-i\theta^x_{mn}} C^\dagger_{m\ n} C_{m+1\ n} \end{aligned}$$

letting $m + 1 = m'$,

$$T^\dagger_x = \sum te^{-i\theta^x_{m'-1\ n}} C^\dagger_{m'-1\ n} C_{m'\ n}.$$

$$\begin{aligned} & H\sum_{m\ n} \psi_{m\ n} C^\dagger_{m\ n}|0\rangle \\ =& \sum_{m\ n} \psi_{m\ n}\left(te^{i\theta^x_{m\ n}} C^\dagger_{m+1\ n} + \ te^{i\theta^y_{m\ n}} C_{m\ n+1} + te^{-i\theta^x_{m-1\ n}} C_{m-1\ n} + \ te^{-i\theta^y_{m\ n-1}} C^\dagger_{m\ n-1}\right)|0\rangle \\ =& \sum_{m\ n} t\left(e^{i\theta^x_{m\ n}} C^\dagger_{m+1\ n} + \ e^{i\theta^y_{m\ n}} C_{m\ n+1} + e^{-i\theta^x_{m-1\ n}} C_{m-1\ n} + \ e^{-i\theta^y_{m\ n-1}} C^\dagger_{m\ n-1}\right) C^\dagger_{m\ n}|0\rangle \\ =& \ E\sum_{m\ n} \psi_{m\ n} C^\dagger_{m\ n}|0\rangle. \end{aligned}$$

$$\therefore t\left(e^{i\theta^x_{m\ n}} C^\dagger_{m+1\ n} + \ e^{i\theta^y_{m\ n}} C_{m\ n+1} + e^{-i\theta^x_{m-1\ n}} C_{m-1\ n} + \ e^{-i\theta^y_{m\ n-1}} C^\dagger_{m\ n-1}\right) = E\psi_{m\ n}.$$

Rotating one plaquette gives

$$\sum_{loop} \theta_{m\ n} := \theta^x_{m\ n} - \theta^x_{m\ n+1} + \theta^y_{m+1\ n} - \theta^y_{m\ n} = 2\pi\phi a^2,$$

where a is a lattice constant.

$$\theta^x_{m\ n} := \frac{2\pi}{\Phi_0}\int_{(m,n)a}^{(m+1,n)a} d\vec{r}\cdot\vec{A},$$

where $\vec{A}$ is a vector potential, $\nabla\times\vec{A}=\vec{B}$. For small a,

$$\theta^x_{m\ n} \simeq \frac{2\pi}{\Phi_0}aA_x,$$

so,

$$\begin{aligned}\sum_{loop}\theta &\simeq -a\partial_y\theta^x + a\partial_x\theta^y\\ &= \frac{2\pi a^2}{\Phi_0}\left(\partial_x A_y - \partial_y A_x\right).\end{aligned}$$

$\left[{}^*\text{cf.}\theta^y \simeq \frac{2\pi}{\Phi_0}aA_y\right]$

$$\begin{aligned}\sum_{loop}\theta &= \frac{2\pi a^2}{\Phi_0}\nabla\times\vec{A}\\ &= \frac{2\pi a^2}{\Phi_0}B = 2\pi\phi,\end{aligned}$$

where $\phi = p/q$, (p,q) mutually prime integers.

$$\phi := \frac{Ba^2}{\Phi_0}\quad \text{Flux per plaquette in unit } \Phi_0.$$

$$\begin{aligned}\psi_{m\ n} &: = \psi\left((m,n)\,a\right),\quad (m,n)=\vec{r}_{mn}\\ \psi_{m+1\ n} &= \psi\left(\vec{r}_{mn}+a\,(1,0)\right)\\ &\simeq \psi\left(\vec{r}_{mn}\right)+a\partial_x\psi+\frac{1}{2}a^2\partial_x^2\psi\end{aligned}$$

$$\begin{aligned}\psi_{m-1\ n} &= \psi\left(\vec{r}_{mn}-a\,(1,0)\right)\\ &\simeq \psi\left(\vec{r}_{mn}\right)-a\partial_x\psi+\frac{1}{2}a^2\partial_x^2\psi\end{aligned}$$

$$t\left(e^{i\theta^x_{m-1\ n}}\psi_{m-1\ n}+e^{-i\theta^x_{m\ n}}\psi_{m\ n}+e^{i\theta^y_{m\ n-1}}\psi_{m\ n-1}+e^{-i\theta^y_{m\ n}}\psi_{m\ n}\right)=E\psi_{m\ n}.$$

Putting Taylor expanded ψ,θ into this equation, we have

$$\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - e\vec{A}\right)^2\psi = \varepsilon\psi,$$

where

$$\varepsilon = \frac{\hbar^2}{2mta^2}\left(E-4t\right).$$

Hamiltonian is

$$H=\sum_{\langle i,j\rangle} te^{i\theta_{ij}}C_i^\dagger C_j + \text{h.c.},$$

we introduce new operator $c'_i := e^{i\phi_i}C_i$, $\left\{c_i, c_j^\dagger\right\} = \delta_{ij}$, $\left\{c'_i, c'^\dagger_j\right\} = \delta_{ij}$,

$$\begin{aligned} H &= \sum_{\langle i,j\rangle} te^{i\theta_{ij}} \left(e^{-i\phi_i}C'_i\right)^\dagger \left(e^{-i\phi_j}C_j\right) \\ &= \sum_{\langle i,j\rangle} te^{i\phi_i}e^{i\theta_{ij}}e^{-i\phi_j}C'^\dagger_i C'_j = e^{i\theta'_{ij}}, \end{aligned}$$

$\theta'_{ij} := \theta_{ij} + \phi_i - \phi_j$.

$$H = \sum_{\langle i,j\rangle} te^{i\theta'_{ij}}C'^\dagger_i C'_j + \text{h.c.}$$

$\left(C'_i, \theta'_{ij}\right)$:gauge symmetry.
*(cf.) Landau gauge and Symmetric gauge.
Take Landau gauge on a lattice

$$2\pi\phi(m+1) - 2\pi\phi m = 2\pi\phi.$$

Uniform $\vec{B}$ on a lattice

$$\phi = \frac{p}{q},$$

where p, q are integer.

$$C_{m\ n} : \text{real space}$$

We take Fourier transformation to momentum space $C\left(\vec{k}\right)$.

$$m = qm' + m''$$

$$C_{m\ n} = \frac{1}{\sqrt{L_x}}\frac{1}{\sqrt{L_y}}\sum_{k_x\ k_y} e^{ik_x m' + ik_y n}C_{m''}(k_x, k_y)$$

$$H = t\sum_{m\ n}\left(C^\dagger_{m+1}C_{m\ n} + \text{h.c.} + e^{i2\pi\phi m}C^\dagger_{m\ n+1}C_{m\ n} + \text{h.c.}\right).$$

Part VII
(2007/10/23)

XI. FOURIER TRANSFORM OF FERMION OPERATORS

$$C_j, \qquad j = 1, 2, 3, \cdots$$

$$\begin{aligned} \{C_i, C_j\} &= 0, \quad \left\{C_i^\dagger, C_j^\dagger\right\} = 0, \quad \left\{C_i, C_j^\dagger\right\} = \delta_{ij}. \\ \{A, B\} &:= AB + BA. \end{aligned}$$

Linear transformation

$$C_j \to \tilde{C}_k, \quad k = 1, 2, \cdots, N.$$

$$\vec{C} = \begin{bmatrix} C_1 \\ \vdots \\ C_N \end{bmatrix}, \quad \vec{\tilde{C}} = \begin{bmatrix} C_1 \\ \vdots \\ C_N \end{bmatrix}.$$

$$\begin{aligned} \vec{C} &= \mathbf{U}\vec{\tilde{C}}, \\ I_N &= \mathbf{U}\mathbf{U}^\dagger = \mathbf{U}^\dagger\mathbf{U}, \quad \mathbf{U} : n \times n \text{ unitary matrix} \end{aligned}$$

For convenient we define $U_{jk} := (\mathbf{U})_{jk}$.

$$C_j = U_{jk} C_k,$$

here we used Einstein convention.

$$\begin{aligned} \mathbf{U}^\dagger \vec{C} &= \mathbf{U}^\dagger \mathbf{U} \vec{\tilde{C}} = \vec{\tilde{C}}. \\ \vec{\tilde{C}}_k &= \left(\mathbf{U}^\dagger\right)_{kj} \vec{C}_j \\ &= U^*_{jk} C_j. \end{aligned}$$

Let us check the commutation relation. It is trivial to show that

$$\left\{\tilde{C}_k, \tilde{C}_{k'}\right\} = \left\{\tilde{C}^\dagger_k, \tilde{C}^\dagger_{k'}\right\} = 0.$$

$$\begin{aligned} \left\{\tilde{C}_k, \tilde{C}^\dagger_{k'}\right\} &= \left\{U^*_{jk} C_j, \left(U^*_{j'k'} C_{j'}\right)^\dagger\right\} \\ &= \left\{U^*_{jk} C_j, U_{j'k'} C^\dagger_{j'}\right\} \\ &= U_{j'k'} U^*_{jk} \left\{C_j, C^\dagger_{j'}\right\} \\ &= U_{j'k'} U^*_{jk} \delta_{j,j'} \\ &= \left(\mathbf{U}^\dagger\right)_{kj} (\mathbf{U})_{jk'} = \delta_{kk'}. \end{aligned}$$

When we take Fourier transformation as a linear transformation,

$$(\mathbf{U})_{jk} = \frac{1}{\sqrt{N}} e^{i\frac{2\pi}{N} k_j},$$

where $\mathbf{U}$ is an unitary matrix.
(*cf) Let us check fact that $\mathbf{U}$ is an unitary matrix.

$$\begin{aligned} \left(\mathbf{U}^\dagger \mathbf{U}\right)_{kk'} &= \left(\mathbf{U}^\dagger\right)_{kj} (\mathbf{U})_{jk'} \\ &= U^*_{jk} U_{jk'} \\ &= \frac{1}{N} e^{-i\frac{2\pi}{N} k_j} e^{-i\frac{2\pi}{N} k'_j} \\ &= \frac{1}{N} \sum_{j=1}^{N} e^{i\frac{2\pi}{N}\left(k'-k\right)j} \\ &= \frac{1}{N} \begin{Bmatrix} N & \text{for } k = k' \\ 0 & \text{for } k \neq k' \end{Bmatrix}. \text{ (Q.E.D.)} \end{aligned}$$

Discrete Fourier transformation

$$K = \frac{2\pi}{N} k,$$

will go continuum limit when $N \to \infty, \Delta K = \frac{2\pi}{N} \to 0$.

Let us consider 1-dimension lattice. N-site problem. (Let N=even number). Let q is a periodic of x-axis. Taking Landau gauge, $\phi = p/q, (p,q) = 1$.

$$C_{m,n} = \frac{1}{\sqrt{L_y}} \sum_{k_y} e^{ik_y n} \frac{1}{\sqrt{L_x/q}} \sum_{k_x} e^{ik_x m'} C_{m''}\left(k_x, k_y\right),$$

where $m = qm' + m'',\ m'' = 1, 2, \cdots, q \quad (m', n):$ label of the cell.

$$H = T_x + T_y + \text{h.c.}$$

$$\begin{aligned} T_x &= \sum t C^\dagger_{m+1} C_{m\ n} \\ &= t \sum_{m\ n} \sum_{k_x k_y} \frac{1}{L_x\ L_y/q} e^{-ik_y n} e^{ik'_y n} e^{-ik_x m'} e^{-ik'_x m'} C^\dagger_{m''+1}\left(k_x, k_y\right) C_{m''}\left(k'_x, k'_y\right) \\ &= t \sum_{m''} C^\dagger_{m''+1}\left(k_x, k_y\right) C_{m''}\left(k_x, k_y\right). \end{aligned}$$

$m'' = 1$ to q.

$$qm' + (q+1) = q\left(m' + 1\right) + 1$$

$$C_{q+1}\left(k_x, k_y\right) = e^{-ik_x} C_1\left(k_x, k_y\right).$$

$$T_x = \begin{bmatrix} C^\dagger_1\left(\vec{k}\right) & C^\dagger_2\left(\vec{k}\right) & \cdots & C^\dagger_q\left(\vec{k}\right) \end{bmatrix} \times \begin{bmatrix} 0 & & & e^{-iqk_x} \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \times \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_q \end{bmatrix}.$$

$$\begin{aligned} T_y &= \sum_{m\ n} t C^\dagger_{m\ n+1} C_{m\ n} \\ &= \sum_{m\ n} \sum_{\substack{k_x\ k_y \\ k'_x, k'_y}} \frac{1}{L_x L_y} e^{-ik_y(n+1)} e^{ik'_y n} e^{ik'_x m'} e^{ik'_x m'} C^\dagger_{m''} C_{m''} \\ &= \sum_{m''} e^{-ik_y} C^\dagger_{m''}\left(\vec{k}\right) C_{m''}\left(\vec{k}\right). \end{aligned}$$

$$T_y = \begin{bmatrix} C^\dagger_1\left(\vec{k}\right) & C^\dagger_2\left(\vec{k}\right) & \cdots & C^\dagger_q\left(\vec{k}\right) \end{bmatrix} \times \begin{bmatrix} e^{-ik_y + 2\pi\phi} & & & 0 \\ & e^{-ik_y 2 + 2\pi\phi 2} & & \\ & & \ddots & \\ 0 & & & e^{-ik_y q + 2\pi\phi q} \end{bmatrix} \times \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_q \end{bmatrix}.$$

$$\begin{aligned} H &= T_x + T_y + \text{h.c.} \\ &= \sum_{\vec{k}} \vec{C}^\dagger\left(\vec{k}\right) \mathbf{H} \vec{C}\left(\vec{k}\right). \end{aligned}$$

$$\begin{aligned} T_y &= \sum C^\dagger_{m\ n+1} C_{m\ n} e^{i2\pi\phi m} \\ &= \sum C^\dagger_{m\ n+1} C_{m\ n} e^{i2\pi\frac{p}{q}\left(qm' + m''\right)} \\ &= \sum C^\dagger_{m\ n+1} C_{m\ n} e^{i2\pi\frac{p}{q} m''} \\ &= \sum C^\dagger_{m\ n+1} C_{m\ n} e^{i2\pi\phi m''}. \end{aligned}$$

$$H = \begin{bmatrix} \cos(k_y - 2\pi\phi\cdot 1) & & & e^{-iqk_x} \\ & \cos(k_y - 2\pi\phi\cdot 2) & & \\ & & \ddots & \\ e^{iqk_x} & & & \cos(k_y - 2\pi\phi\cdot q) \end{bmatrix}.$$

$$\begin{aligned} \vec{k} &= (k_x, k_y) \quad : \text{Brillouin zone} \\ k_x &\in [0, 2\pi], \quad k_y \in [0, 2\pi] \\ L_x, L_y &\to \infty. \\ C\left(\vec{k} + (2\pi, 0)\right) &= C\left(\vec{k} + (0, 2\pi)\right) = C\left(\vec{k}\right). \end{aligned}$$

Energy eigenvalue ?

$$H = \sum_{\vec{k}} \sigma^\dagger\left(\vec{k}\right)\mathbf{H}\left(\vec{k}\right)\sigma\left(\vec{k}\right),$$

where $\mathbf{H}$ is an hermite matrix. It diagonalized by unitary matrix.

$$\begin{aligned} \mathbf{H}\phi_1 &= E_1\phi_1, \\ &\vdots \\ \mathbf{H}\phi_q &= E_q\phi_q, \end{aligned}$$

here, we put $E_i \leq E_j \ \ (i < j)$, $\phi_j^\dagger\phi_{j'} = \delta_{jj'}$ (orthonormalized ϕ.) .

$$\mathbf{HU} = \mathbf{DU},$$

where

$$\begin{aligned} \mathbf{U} &= \left[\vec{\phi}_1 \ \ \vec{\phi}_2 \ \ \cdots \ \ \vec{\phi}_q\right], \\ \mathbf{D} &= \begin{bmatrix} E_1 & & & 0 \\ & E_2 & & \\ & & \ddots & \\ 0 & & & E_q \end{bmatrix}. \end{aligned}$$

E_j?

$$E_j\left(\vec{k}\right) \ : \text{one particle energy.}$$

$$\begin{aligned} H|\phi_j\rangle &= E_j|\phi_j\rangle, \\ H &= \sum_{\vec{k}} \vec{C}^\dagger\left(\vec{k}\right)\mathbf{H}\left(\vec{k}\right)C\left(\vec{k}\right). \end{aligned}$$

$$\begin{aligned} |\vec{\phi}_j\rangle &= \vec{C}^\dagger\vec{\phi}|0\rangle \\ |\vec{\phi}_j\rangle &= |\phi_j\left(\vec{k}\right)\rangle = \left[C_1^\dagger\left(\vec{k}\right) \ \ C_2^\dagger\left(\vec{k}\right) \ \ \cdots \ \ C_q^\dagger\left(\vec{k}\right)\right] \times \begin{bmatrix} (\phi_j)_1 \\ (\phi_j)_2 \\ \vdots \\ (\phi_j)_q \end{bmatrix} |0\rangle \\ &= C_l^\dagger\left(\vec{k}\right)\phi_l|0\rangle. \end{aligned}$$

$$H|\vec{\phi}_j\left(\vec{k}\right)\rangle = \left(\sum_{k'} \vec{C}^\dagger(k')\,\mathbf{H}(k')\,\vec{C}(k')\right)\vec{C}^\dagger\left(\vec{k}\right)\vec{\phi}|0\rangle.$$

$\left\{C_j(k), C_{j'}^{\dagger}(k')\right\} = \delta_{jj'}\delta_{\vec{k}\vec{k}'}$.

$$\begin{aligned}
H|\phi_j\rangle &= \sum_{k'} C^{\dagger}\left(\vec{k}\right)\mathbf{H}(k')\begin{bmatrix} C_1(k') \\ C_2(k') \\ \vdots \\ C_q(k')\end{bmatrix}\left[C_1^{\dagger}\left(\vec{k}\right) \;\; C_2^{\dagger}\left(\vec{k}\right) \;\; \cdots \;\; C_q^{\dagger}\left(\vec{k}\right)\right]\vec{\phi}|0\rangle \\
&= C^{\dagger}(k)\mathbf{H}(k')\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1\end{bmatrix}|0\rangle\;\vec{\phi}_j \\
&= C^{\dagger}(k)\mathbf{H}(k')\vec{\phi}_j|0\rangle \\
&= E_j\; C^{\dagger}(k)\vec{\phi}_j|0\rangle.
\end{aligned}$$

For many particle state (Fermi sea.)

$$|G\rangle = \prod_{k,j}\left[\vec{C}_j^{\dagger}(k)\vec{\phi}_j\right]|0\rangle,$$

$E_j(k) \le E_F,\; E_j(k) \quad j = 1, 2, \cdots, q. \;\; \phi = p/q,\; \phi \in [0,1]$. We can find Hofstater's Butterfly.
Adiabatic approximation ($T \to \infty$)
Linear response $\to$ Hall current.
σ_{xy} $\to$topological integers.

Part VIII
(2007/11/06)

XII. QHE

For 2-dimensional metal plane with z-directional magnetic field and x directional electric potential, we have seen

$$I_y = -\frac{\partial H}{\partial A_y}.$$

For an infinitesimal translation, $\theta^y = 2\pi\phi m + 2\pi\frac{1}{\Phi_0}aA_y$.The Hamiltonian is

$$\begin{aligned}
H &= \sum C_{m\;n+1}^{\dagger}e^{i\;2\pi\phi m + i2\pi\frac{1}{\Phi_0}aA_y}C_{m\;n} + \cdots, \\
C_{m,n} &= \frac{1}{\sqrt{L_y}}\sum_{k_y}e^{ik_y n}\frac{1}{\sqrt{L_x/q}}\sum_{k_x}e^{ik_x m'}C_{m''}(k_x, k_y).
\end{aligned}$$

For

$$-ik_y + i2\pi\frac{1}{\Phi_0}aA_y = -i\left(k_y - 2\pi\frac{aA_y}{\Phi_0}\right) = 0,$$

we have

$$\begin{aligned}
\left(-2\pi\frac{a}{\Phi_0}\right)\frac{\partial}{\partial k_y} &= \frac{\partial}{\partial A_y}, \\
\hat{I}_y &= -\frac{\partial \hat{H}}{\partial A_y} = 2\pi\frac{a}{\Phi_0}\frac{\partial H}{\partial k_y}.
\end{aligned}$$

$$I_y = \langle G'|\hat{I}_y|G'\rangle,$$

here, we use the linear response theory, $|G'\rangle = |G\rangle + V_x$.

$$I_y = \sigma_{yx} E_x,$$

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t},$$

let $\phi = 0$, and $\vec{A}$ change slowly, then we see

$$A_x = -E_x t.$$

$$k_x \to k_x + 2\pi \frac{a}{\Phi_0} E_x t.$$

For $|G'\rangle$,

$$H\left(k_x + 2\pi \frac{aE_x t}{\Phi_0}, k_y - 2\pi \frac{aA_y}{\Phi_0}\right).$$

For $|G\rangle$,

$$H\left(k_x, k_y - 2\pi \frac{aA_y}{\Phi_0}\right).$$

$$i\hbar|G'(t)\rangle = H(t)|G'(t)\rangle \;\; : \text{time-dependent Schrodinger equation.}$$

$$H(t)|\alpha(t)\rangle = E_\alpha(t)|\alpha(t)\rangle, \quad \alpha = 1, 2, 3, \cdots$$

let g is the $\alpha = 1$ state(ground state).

$$|G'\rangle = e^{\frac{1}{i\hbar}\int_0^t dt' E_g(t')} \sum_\alpha \dot{a}_\alpha(t)|\alpha(t)\rangle,$$

then,

$$\begin{aligned} i\hbar\partial_t|G'\rangle &= E_g|G'\rangle + e^{\frac{1}{i\hbar}\int_0^t dt' E_g(t')} \sum_\alpha \left(\dot{a}_\alpha(t)|\alpha(t)\rangle + a_\alpha(t)|\dot{\alpha}(t)\rangle\right) \\ &= He^{\frac{1}{i\hbar}\int_0^t dt' E_g(t')} \sum_\alpha \left(a_\alpha(t)|\alpha(t)\rangle\right) \\ &= e^{\frac{1}{i\hbar}\int_0^t dt' E_g(t')} \sum_\alpha \left(a_\alpha(t) E_\alpha|\alpha(t)\rangle\right). \end{aligned} \tag{6}$$

Taking $\langle g|$ to (Eq. 6),

$$\begin{aligned} a_g E_g &= E_g a_g + \dot{a}_g + \sum_\alpha a_\alpha \langle g|\dot{\alpha}\rangle, \\ \dot{a}_g &= -\sum_\alpha a_\alpha \langle g|\dot{\alpha}\rangle. \end{aligned}$$

For small $t, |G'\rangle \sim |g\rangle$,

$$\begin{aligned} \dot{a}_g &\simeq -a_g\langle g|\dot{g}\rangle, \\ a_g &\simeq e^{-\int^t \langle g|\dot{g}\rangle dt}, \quad \text{Berry phase.} \end{aligned}$$

For $\alpha \neq g$, taking $\langle\alpha|$ to (Eq. 6),

$$a_\alpha E_\alpha = E_g a_g + \dot{a}_g + \sum_{\alpha'} a_{\alpha'} \langle\alpha|\dot{\alpha}'\rangle,$$

for small t,

$$(E_\alpha - E_g)\, a_\alpha \simeq \dot{a}_\alpha + a_g \langle \alpha | \dot{g} \rangle$$

if we neglect $\dot{a}_\alpha$,

$$a_\alpha = \frac{\langle \alpha | \dot{g} \rangle}{E_\alpha - E_g} a_g.$$

(Adiabatic approximation.)

Part IX
(2007/11/13)

$$\begin{aligned} |G\rangle &= \sum_\alpha C_\alpha |\alpha(t)\rangle \\ |\alpha(t)\rangle &: \text{ eigen state of the snap shot Hamiltonian} \end{aligned}$$

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} |G\rangle &= H |G\rangle \\ H(t)\, |\alpha(t)\rangle &= E_\alpha(t)\, |\alpha(t)\rangle \end{aligned}$$

$$\begin{aligned} C_g &\approx e^{i\gamma}, \\ \text{where } \gamma &= -\int_0^t dt \langle g | \dot{g} \rangle \quad \text{Berry phase} \end{aligned}$$

$$\begin{aligned} C_\alpha &\approx i\hbar C_g \frac{\langle \alpha | \dot{g} \rangle}{E_g - E_\alpha} \\ |C_\alpha| &\ll |C_g| \quad (\alpha \neq g) \end{aligned}$$

$$\begin{aligned} \langle I_y \rangle &= \langle G | I_y | G \rangle - \langle g | I_y | g \rangle \\ &= \left(C_g^* \langle g | + \sum_{\alpha \neq g} C_\alpha^* \langle \alpha | \right) I_y \left(C_g |g\rangle + \sum_{\alpha \neq g} C_\alpha |\alpha\rangle \right) - \langle g | I_y | g \rangle \end{aligned}$$

Using $C_g^* C_g = 1,\ |C_\alpha| \ll |C_g|\ \ (\alpha \neq g)$,

$$\begin{aligned}
\langle I_y \rangle &= \sum_{\alpha\neq g} \left(C_g^* \langle g|I_y|\alpha\rangle C_\alpha + C_\alpha^* \langle \alpha|I_y|g\rangle C_g \right) \\
&\approx \sum_{\alpha\neq g} \left(i\hbar \frac{\langle g|I_y|\alpha\rangle\langle\alpha|\dot{g}\rangle}{E_g - E_\alpha} - i\hbar \frac{\langle\dot{g}|\alpha\rangle\langle\alpha|I_y|g\rangle}{E_g - E_\alpha} \right), \\
&\quad \text{using } \left\{ I_y = 2\pi \frac{a}{\Phi_0} \frac{\partial H}{\partial k_y} \right\}, \\
&= i\hbar 2\pi \frac{a}{\Phi_0} \sum_{\alpha\neq g} \left(i\hbar \frac{\langle g|\frac{\partial H}{\partial k_y}|\alpha\rangle\langle\alpha|\dot{g}\rangle}{E_g - E_\alpha} - i\hbar \frac{\langle\dot{g}|\alpha\rangle\langle\alpha|\frac{\partial H}{\partial k_y}|g\rangle}{E_g - E_\alpha} \right), \\
&\quad \text{using } \left\{ \frac{\partial}{\partial t} = 2\pi \frac{aE_x}{\Phi_0} \frac{\partial}{\partial k_y} \right\}, \text{let } \left\{ \frac{\partial}{\partial k_y} =: \partial_y \right\} \\
&= i\hbar 2\pi \frac{a}{\Phi_0} 2\pi \frac{aE_x}{\Phi_0} \sum_{\alpha\neq g} \left(i\hbar \frac{\langle g|\frac{\partial H}{\partial k_y}|\alpha\rangle\langle\alpha|\partial_x g\rangle}{E_g - E_\alpha} - i\hbar \frac{\langle\partial_x g|\alpha\rangle\langle\alpha|\frac{\partial H}{\partial k_y}|g\rangle}{E_g - E_\alpha} \right), \\
&\quad \text{using } \left\{ \hbar = \frac{h}{2\pi}, \Phi_0 = \frac{h}{e} \right\}, \\
&= \frac{e^2}{h} i a^2 E_x 2\pi \sum_{\alpha\neq g} \left(i\hbar \frac{\langle g|\frac{\partial H}{\partial k_y}|\alpha\rangle\langle\alpha|\partial_x g\rangle}{E_g - E_\alpha} - i\hbar \frac{\langle\partial_x g|\alpha\rangle\langle\alpha|\frac{\partial H}{\partial k_y}|g\rangle}{E_g - E_\alpha} \right). \qquad (7)
\end{aligned}$$

Using

$$\begin{aligned}
H|\alpha\rangle &= E_\alpha|\alpha\rangle \\
\partial_y \left(H|g\rangle \right) &= \partial_y \left(E_g|g\rangle \right) \\
\left(\partial_y H \right)|g\rangle + H|\partial_y g\rangle &= \left(\partial_y E_g \right)|g\rangle + E_\alpha|\partial_y g\rangle
\end{aligned}$$

$$\begin{aligned}
&\therefore\ \langle\alpha|\partial_y H|g\rangle + E_\alpha\langle\alpha|\partial_y g\rangle = \langle\alpha|g\rangle\partial_y E_g + E_g\langle\alpha|\partial_y g\rangle \\
&\therefore\ \langle\alpha|\partial_y H|g\rangle = \left(E_g - E_\alpha \right)\langle\alpha|\partial_y g\rangle,
\end{aligned}$$

(Eq. 7) become

$$\begin{aligned}
&\frac{e^2}{h} i a^2 E_x 2\pi \sum_{\alpha\neq g} \left(\langle\partial_x g|\alpha\rangle\langle\alpha|\partial_y g\rangle - \langle\partial_y g|\alpha\rangle\langle\alpha|\partial_x g\rangle \right), \\
&\text{using } \left\{ \sum_\alpha |\alpha\rangle\langle\alpha| \approx \sum_{\alpha\neq g} |\alpha\rangle\langle\alpha| = 1 \right\}, \\
= &\frac{e^2}{h} i a^2 E_x 2\pi \left(\langle\partial_x g|\partial_y g\rangle - \langle\partial_y g|\partial_x g\rangle \right). \qquad (8)
\end{aligned}$$

Using

$$\begin{aligned}
|g\rangle &= \prod_{\substack{l \\ E_l(\vec{k})<E_F}} C^\dagger \psi_l(k)|0\rangle, \\
|\partial_x g\rangle &= \sum_{k_x} \cdots |0\rangle, \\
\langle\partial_x g|\partial_y g\rangle &= \sum_{\varepsilon_l(\vec{k})<E_F} \left(\partial_x \psi_l^\dagger \partial_y \psi_l - \partial_y \psi_l^\dagger \partial_x \psi_l \right),
\end{aligned}$$

(Eq. 8) becomes

$$\begin{aligned}
& \frac{e^2}{h} i E_x 2\pi L_x L_y \int \frac{d^2k}{(2\pi)^2} \sum_i \left(\partial_x \psi_l^\dagger \partial_y \psi - \partial_y \psi_l^\dagger \partial_x \psi_l \right) \\
= & \frac{e^2}{h} i E_x 2\pi L_x L_y \int \frac{d^2k}{(2\pi)^2} \sum_i \left(\nabla \times \vec{A} \right)_z \\
= & \frac{e^2}{h} i V_x L_x L_y \frac{-1}{2\pi i} \int d^2k \sum_i \left(\nabla \times \vec{A} \right)_z \\
= & j_y L_y .
\end{aligned}$$

$$\begin{aligned}
j_y &= \sigma_{yx} V_x \\
\sigma_{xy} &= -\frac{e^2}{h} \frac{1}{2\pi i} \int dk^2 \left(\nabla \times \vec{A} \right)_z .
\end{aligned}$$

Part X
(2007/11/20)

$$\begin{aligned}
\sigma_{xy} &= \frac{e^2}{h} C \\
C &= \frac{1}{2\pi i} \sum_j \int_{\varepsilon_j(k) < E_F(k)} d^2k \left(\partial_x \vec{\psi}_j^\dagger \partial_y \vec{\psi}_j - \partial_y \vec{\psi}_j^\dagger \partial_x \vec{\psi}_j \right) \\
&= \sum_{j=1 \sim N_F} C_j \\
\psi &: \text{one particle wave function} \\
N_F &: \text{number of bands blow } E_F .
\end{aligned}$$

$$\begin{aligned}
H\left(\vec{k}\right) \vec{\psi}_j \left(\vec{k}\right) &= E_j \left(\vec{k}\right) \vec{\psi}_j \left(\vec{k}\right) \\
H &: q \times q \text{ matrix} \\
\psi_i &= \begin{bmatrix} \psi_j^1 \\ \vdots \\ \psi_j^q \end{bmatrix} \\
H|\psi_j\rangle &= E_j |\psi_j\rangle
\end{aligned}$$

$$C_j = \frac{1}{2\pi i} \int_{\varepsilon_j(k) < E_F(k)} d^2k \left(\langle \partial_x \psi | \partial_y \psi \rangle - \langle \partial_y \psi | \partial_x \psi \rangle \right)$$

$$\begin{aligned}
\vec{A}_j &= \langle \psi_i | \nabla \psi_j \rangle \\
A^x &= \langle \psi | \nabla \psi \rangle = \langle \psi | \partial_x \psi \rangle \quad (j \text{ is omitted}) \\
A^y &= \langle \psi | \nabla \psi \rangle = \langle \psi | \partial_y \psi \rangle
\end{aligned}$$

$$\begin{aligned}
\left(\nabla \times \vec{A} \right)_z &= \partial_x A_y - \partial_y A_x \\
&= \partial_x \left(\langle \psi | \partial_y \psi \rangle \right) - \partial_y \left(\langle \psi | \partial_x \psi \rangle \right) \\
&= \langle \partial_x \psi | \partial_y \psi \rangle + \langle \psi | \partial_x \partial_y \psi \rangle - \langle \partial_y \psi | \partial_x \psi \rangle - \langle \psi | \partial_y \partial_x \psi \rangle \\
&= \langle \partial_x \psi | \partial_y \psi \rangle - \langle \partial_y \psi | \partial_x \psi \rangle .
\end{aligned}$$

Charm number of the j-th state is given by

$$C_j = \frac{1}{2\pi i}\int_{BZ} d^2k\,(\nabla\times A)_z = \frac{1}{2\pi i}\oint_{\partial T^2=0} d\vec{l}\cdot A.$$

Is this value zero ?

$$\begin{aligned}\vec{A} &= \langle\psi|\nabla_k\psi\rangle \\ H\left(\vec{k}\right)|\psi(k)\rangle e^{i\theta} &= E(k)\,|\psi(k)\rangle e^{i\theta} \\ &\quad \text{each } k \text{ is independent}\end{aligned}$$

$$H|\psi\rangle = E|\psi\rangle.$$

Let $e^{i\theta} = \omega$,

$$\begin{aligned}H|\psi\rangle\omega &= E|\psi\rangle\omega \\ H|\psi'\rangle &= E|\psi'\rangle \\ \text{where } |\psi'\rangle &= |\psi\rangle\omega \\ \langle\psi|\psi\rangle &= 1 \\ \langle\psi'|\psi'\rangle &= |\omega|^2\end{aligned}$$

$$\therefore |\omega|^2 = 1$$

ω is smooth in $\vec{k}$.

$$\begin{aligned}\nabla|\psi'\rangle &= \nabla\left(|\psi\rangle\omega\right) \\ &= |\nabla\psi\rangle\omega + |\psi\rangle\left(\nabla\omega\right)\end{aligned}$$

$$A' = \omega^* A\omega + \omega^*\nabla\omega$$

$$\begin{aligned}\omega &= e^{i\theta} \\ A' &= A\,|\omega|^2 + e^{-i\theta}\left(i\nabla\theta\right)e^{i\theta} \\ &= A + i\nabla\theta \quad : \text{gauge transformation in k space.}\end{aligned}$$

$$\begin{aligned}\nabla\times\vec{A}' &= \nabla\times\vec{A}' \\ &= \nabla\times\left(\vec{A} + i\nabla\theta\right) \\ &= \nabla\times\vec{A}.\end{aligned}$$

C is gauge invariant.

$$\text{choice of } \omega = e^{i\theta} \Leftrightarrow \text{choice of gauge}$$

$$\begin{aligned}P &= |\psi\rangle\langle\psi| \\ P^2 &= P \\ P' &= |\psi'\rangle\langle\psi'| \\ &= |\psi\rangle\langle\psi|\ \ |\omega|^2 \\ &= P.\end{aligned}$$

P is also gauge invariant.

$$\begin{aligned}
\exists\ |\phi\rangle &\ \ \text{fixed state} \\
|\psi^U\rangle &= P|\phi\rangle\ \ : \text{gauge fixing}
\end{aligned}$$

$$\begin{aligned}
H|\psi^U\rangle &= HP|\phi\rangle \\
&= H|\psi\rangle\langle\psi|\phi\rangle \\
&= E|\psi\rangle\langle\psi|\phi\rangle \\
&= E|\psi^U\rangle .
\end{aligned}$$

$|\psi^U\rangle$ is normalized ? Generically net normalized

$$\begin{aligned}
N &= \langle\psi^U|\psi^U\rangle \\
&= \langle\phi|P\cdot P|\phi\rangle \\
&= \langle\phi|P|\phi\rangle \\
&= \langle\phi|\psi\rangle\langle\psi|\phi\rangle \\
&= |\langle\psi|\phi\rangle|^2 .
\end{aligned}$$

$$\begin{aligned}
N' &= \left|\langle\psi'|\phi\rangle\right|^2 \\
&= \left|\omega^*\langle\psi'|\phi\rangle\right|^2 \\
&= |\langle\psi|\phi\rangle|^2 = N\ \ : \text{gauge invariant } N \neq 0
\end{aligned}$$

$$\begin{aligned}
|\psi^N\rangle &= \frac{1}{\sqrt{N}}|\psi^N\rangle . \\
\langle\psi^N|\psi^N\rangle &= 1.
\end{aligned}$$

When $N \neq 0$ for $\forall\vec{k}$, $C_j = 0$.

$$\begin{aligned}
T^2 &= D_< \cup D_> \\
&\ \ \ D_< \text{ inside of } T^2 \text{ torus} \\
&\ \ \ D_> \text{ outside of } T^2 \text{ torus}
\end{aligned}$$

$$\begin{aligned}
C_j &= \frac{1}{2\pi i}\int_{D_<\cup D_>} d^2k\left(\nabla\times\vec{A}\right) \\
&= \frac{1}{2\pi i}\int_{D_<} d^2k\left(\nabla\times\vec{A}'\right) + \frac{1}{2\pi i}\int_{D_>} d^2k\left(\nabla\times\vec{A}\right) \\
&= \frac{1}{2\pi i}\oint_{\partial_{D_<}} d\vec{k}\cdot\vec{A}' - \frac{1}{2\pi i}\oint_{\partial_{D_<}} d\vec{k}\cdot\vec{A} \\
&= \frac{1}{2\pi i}\oint_{\partial_{D_<}} d\vec{k}\cdot\left(\vec{A}' - \vec{A}\right)
\end{aligned}$$

$A \leftrightarrow A'$

$$\begin{aligned}
|\psi^N\rangle &= \frac{1}{\sqrt{N}}P|\phi\rangle = \frac{1}{\sqrt{N}}|\psi\rangle\langle\psi|\phi\rangle \\
|\psi^N\rangle &= \frac{1}{\sqrt{N}}P|\phi'\rangle = \frac{1}{\sqrt{N}}|\psi\rangle\langle\psi|\phi'\rangle = |\psi^N\rangle\omega,
\end{aligned}$$

$$
\begin{aligned}
\omega &= \frac{\sqrt{N}\langle\psi|\phi'\rangle}{\sqrt{N'}\langle\psi|\phi\rangle} \\
&= \frac{|\langle\psi|\phi\rangle|\,|\langle\psi|\phi\rangle|\,e^{iArg\langle\psi|\phi'\rangle}}{|\langle\psi|\phi\rangle|\,|\langle\psi|\phi\rangle|\,e^{iArg\langle\psi|\phi\rangle}} \\
&= e^{iArg\frac{\langle\psi|\phi'\rangle}{\langle\psi|\phi\rangle}} = e^{i\theta}, \\
\theta &= Arg\frac{\langle\psi|\phi'\rangle}{\langle\psi|\phi\rangle}.
\end{aligned}
$$

$$
\begin{aligned}
A' &= \omega^*\langle\psi^N|\nabla\psi^N\rangle\omega + \omega^*\langle\psi^N|\psi^N\rangle\nabla\omega \\
&= A + \omega^*\nabla\omega \\
&= A + e^{-i\theta}\left(i\nabla\theta\right)e^{i\theta} \\
&= A + i\nabla\theta.
\end{aligned}
$$

$$
\begin{aligned}
C_j &= \frac{1}{2\pi i}\oint_{\partial D_<} d\vec{k}\cdot\left(\vec{A}'-\vec{A}\right) \\
&= \frac{1}{2\pi i}\oint_{\partial D<} d\vec{k}\cdot(i\nabla\theta) \\
&= \frac{1}{2\pi}\oint_{\partial D<} d\vec{k}\cdot(\nabla\theta)
\end{aligned}
$$

Assumption $\frac{\langle\psi|\phi'\rangle}{\langle\psi|\phi\rangle}$ single valued on ∂D

$$
\begin{aligned}
C_j &= \frac{1}{2\pi}2\pi n = n \in \mathbb{Z} \\
\sigma_{xy} &= \frac{e^2}{\hbar}C \\
C &= \sum_j C_j = \sum_j n_j \\
\sigma_{xy} &: \text{ intrinsically integer}
\end{aligned}
$$

$$
\vec{A} = \langle\psi|\nabla\psi\rangle
$$

Berry's connection

$$
\begin{aligned}
H(R)\,|\psi(R)\rangle &= E(R)\,|\psi(R)\rangle \\
R &: \text{ any parameter } R = \left(R^1, R^2, \cdots, R^D\right)
\end{aligned}
$$

$$
A_\mu = \langle\psi|\partial_\mu\psi\rangle \quad \mu = 1, 2, \cdots D, \quad D\text{: \# of parameter}
$$

$$
\partial_\mu = \frac{\partial}{\partial R_\mu}
$$

$$
\begin{aligned}
i\nu_c &= \oint dR^\mu A_\mu \quad : \text{ Berry phase} \\
&= \oint_c dR^\mu\langle\psi|\partial_\mu\psi\rangle
\end{aligned}
$$