09/20/2011

ハミルトニアン

$$\hat{H} - t\sum_{<i,j>} \hat{c}^{\dagger}_{\circ i}\hat{c}_{\bullet j} + \text{h.c.} \tag{1}$$

を考える。

$$\hat{\boldsymbol{c}}_{\circ} \overset{\text{def}}{=} \begin{pmatrix} \hat{c}_{\circ 1} \\ \vdots \\ \hat{c}_{\circ n} \end{pmatrix} \tag{2}$$

$$\hat{\boldsymbol{c}}_{\bullet} \overset{\text{def}}{=} \begin{pmatrix} \hat{c}_{\bullet 1} \\ \vdots \\ \hat{c}_{\bullet n} \end{pmatrix} \tag{3}$$

$$\boldsymbol{H} \overset{\text{def}}{=} \begin{pmatrix} \boldsymbol{O} & \boldsymbol{D} \\ \boldsymbol{D}^t & \boldsymbol{O} \end{pmatrix} \tag{4}$$

とすると (1) は

$$\hat{H} = \begin{pmatrix} \hat{\boldsymbol{c}}^{\dagger}_{\bullet} & \hat{\boldsymbol{c}}^{\dagger}_{\circ} \end{pmatrix} \boldsymbol{H} \begin{pmatrix} \hat{\boldsymbol{c}}_{\bullet} \\ \hat{\boldsymbol{c}}_{\circ} \end{pmatrix} \tag{5}$$

と表せる。

$$\hat{\boldsymbol{c}}_{\bullet} \to \hat{\boldsymbol{c}}_{\bullet} \overset{\text{def}}{=} \hat{\boldsymbol{d}}_{\bullet} \tag{6}$$

$$\hat{\boldsymbol{c}}_{\circ} \to -\hat{\boldsymbol{c}}_{\circ} \overset{\text{def}}{=} \hat{\boldsymbol{d}}_{\circ} \tag{7}$$

を導入すると

$$\begin{pmatrix} \hat{\boldsymbol{c}}_{\bullet} \\ \hat{\boldsymbol{c}}_{\circ} \end{pmatrix} = \begin{pmatrix} \hat{\boldsymbol{d}}_{\bullet} \\ -\hat{\boldsymbol{d}}_{\circ} \end{pmatrix} \tag{8}$$

$$= \begin{pmatrix} \boldsymbol{I} & \boldsymbol{O} \\ \boldsymbol{O} & -\boldsymbol{I} \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{d}}_{\bullet} \\ \hat{\boldsymbol{d}}_{\circ} \end{pmatrix} \tag{9}$$

$$\overset{\text{def}}{=} \boldsymbol{\Gamma} \begin{pmatrix} \hat{\boldsymbol{d}}_{\bullet} \\ \hat{\boldsymbol{d}}_{\circ} \end{pmatrix} \tag{10}$$

これより

$$\boldsymbol{\Gamma}^{\dagger}\boldsymbol{H}\boldsymbol{\Gamma} = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{O} \\ \boldsymbol{O} & -\boldsymbol{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{O} & \boldsymbol{D} \\ \boldsymbol{D}^{\dagger} & \boldsymbol{O} \end{pmatrix} \begin{pmatrix} \boldsymbol{I} & \boldsymbol{O} \\ \boldsymbol{O} & -\boldsymbol{I} \end{pmatrix} \tag{11}$$

$$= \begin{pmatrix} \boldsymbol{O} & \boldsymbol{D} \\ -\boldsymbol{D}^{\dagger} & \boldsymbol{O} \end{pmatrix} \begin{pmatrix} \boldsymbol{I} & \boldsymbol{O} \\ \boldsymbol{O} & -\boldsymbol{I} \end{pmatrix} \tag{12}$$

$$= \begin{pmatrix} \boldsymbol{O} & -\boldsymbol{D} \\ -\boldsymbol{D}^{\dagger} & \boldsymbol{O} \end{pmatrix} \tag{13}$$

$$= -\boldsymbol{H} \tag{14}$$

よって

$$\{\boldsymbol{H}, \boldsymbol{\Gamma}\} = \boldsymbol{H}\boldsymbol{\Gamma} + \boldsymbol{\Gamma}\boldsymbol{H} = \boldsymbol{O} \tag{15}$$

次の変換によって運動量空間に移る。

$$\hat{c}_{\bullet i} = \frac{1}{\sqrt{N}} \sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}_i} \hat{c}_{\bullet}(\boldsymbol{k}) \tag{16}$$

$$\hat{c}_{\circ i} = \frac{1}{\sqrt{N}} \sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}_i} \hat{c}_{\circ}(\boldsymbol{k}) \tag{17}$$

$$\Delta(\boldsymbol{k}) = -t(1 + e^{ik_1} + e^{ik_2}) \tag{18}$$

これを用いると $\hat{H}$ は

$$\hat{H} = \sum_{\boldsymbol{k}} \hat{c}^{\dagger}_{\bullet}(\boldsymbol{k})\Delta(\boldsymbol{k})\hat{c}_{\circ}(\boldsymbol{k}) + \text{h.c.} \tag{19}$$

$$= \sum_{\boldsymbol{k}} \begin{pmatrix} \hat{c}^{\dagger}_{\bullet}(\boldsymbol{k}) & \hat{c}^{\dagger}_{\circ}(\boldsymbol{k}) \end{pmatrix} \begin{pmatrix} 0 & \Delta(\boldsymbol{k}) \\ \Delta^*(\boldsymbol{k}) & 0 \end{pmatrix} \begin{pmatrix} \hat{c}_{\bullet}(\boldsymbol{k}) \\ \hat{c}_{\circ}(\boldsymbol{k}) \end{pmatrix} \tag{20}$$

となる。よって

$$h(\boldsymbol{k}) = \begin{pmatrix} 0 & \Delta(\boldsymbol{k}) \\ \Delta^*(\boldsymbol{k}) & 0 \end{pmatrix} \tag{21}$$

の固有値は $E_1 = +|\Delta(\boldsymbol{k})|, E_2 = -|\Delta(\boldsymbol{k})|$ である。

$$\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{22}$$

に対して

$$\{h, \Gamma\} = h\Gamma + \Gamma h = 0 \Longrightarrow \Gamma h \Gamma + h = 0 \tag{23}$$

が成り立つ。2×2 の行列 h を Pauli 行列で展開する。

$$h = R_x\sigma_x + R_y\sigma_y + R_z\sigma_z \tag{24}$$

$$= R_\mu\sigma_\mu \tag{25}$$

Pauli 行列

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{26}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{27}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{28}$$

$$\tag{29}$$

は次の性質をもつ。

$$\sigma_i^2 = 1 \tag{30}$$

$$\sigma_i\sigma_j = \epsilon_{ijk}\sigma_k \qquad \begin{pmatrix} \sigma_x\sigma_y = i\sigma_z \\ \sigma_y\sigma_z = i\sigma_x \\ \sigma_z\sigma_x = i\sigma_y \end{pmatrix} \tag{31}$$

ここで

$$\epsilon_{ijk} = \begin{cases} 1 & (i,j,k) = (1,2,3),(2,3,1),(3,1,2) \\ -1 & (i,j,k) = (1,3,2),(3,2,1),(2,1,3) \\ 0 & \text{otherwise} \end{cases} \tag{32}$$

また

$$h^{\dagger} = R_i^*\sigma_i = r_i\sigma_i = h \qquad R_i \in \mathbb{R} \tag{33}$$

であるので

$$h = \boldsymbol{R}\cdot\boldsymbol{\sigma} \tag{34}$$

数学公式

$$(\boldsymbol{A}\cdot\boldsymbol{\sigma})(\boldsymbol{B}\cdot\boldsymbol{\sigma}) = (\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{I} + \mathrm{i}(\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{\sigma} \tag{35}$$

を使うと

$$\boldsymbol{h}^2 = (\boldsymbol{R}\cdot\boldsymbol{\sigma})(\boldsymbol{R}\cdot\boldsymbol{\sigma}) \tag{36}$$
$$= (\boldsymbol{R}\cdot\boldsymbol{R})\boldsymbol{I} + \mathrm{i}(\boldsymbol{R}\times\boldsymbol{R})\cdot\boldsymbol{\sigma} \tag{37}$$
$$= |\boldsymbol{R}|^2\boldsymbol{I} \tag{38}$$

したがって

$$|\mathit{\Delta}|^2 = |\boldsymbol{R}|^2 \tag{39}$$

つまりエネルギーギャップは $2|\boldsymbol{R}|$。このギャップが $\boldsymbol{k}_0$ で閉じるとし、$\boldsymbol{h}$ を $\boldsymbol{k}_0$ の近傍で展開する。

$$\boldsymbol{h} = \boldsymbol{R}\cdot\boldsymbol{\sigma} \tag{40}$$
$$\approx \boldsymbol{R}(\boldsymbol{k}_0) + \delta k_x \left.\frac{\partial\boldsymbol{R}}{\partial k_x}\right|_{\boldsymbol{k}=\boldsymbol{k}_0}\cdot\boldsymbol{\sigma} + \delta k_y \left.\frac{\partial\boldsymbol{R}}{\partial k_y}\right|_{\boldsymbol{k}=\boldsymbol{k}_0}\cdot\boldsymbol{\sigma} \tag{41}$$
$$= (\boldsymbol{X}\cdot\boldsymbol{\sigma})\delta k_x + (\boldsymbol{Y}\cdot\boldsymbol{\sigma})\delta k_y \tag{42}$$
$$= \left(\boldsymbol{X}\delta k_x + \boldsymbol{Y}\delta k_y\right)\cdot\boldsymbol{\sigma} \tag{43}$$

ここで

$$\delta k_x \overset{\mathrm{def}}{=} k_x - k_{0x} \tag{44}$$
$$\delta k_y \overset{\mathrm{def}}{=} k_y - k_{0y} \tag{45}$$
$$\boldsymbol{X} \overset{\mathrm{def}}{=} \frac{\partial\boldsymbol{R}}{\partial k_x} \tag{46}$$
$$\boldsymbol{Y} \overset{\mathrm{def}}{=} \frac{\partial\boldsymbol{R}}{\partial k_y} \tag{47}$$

である。

$$\boldsymbol{\Gamma} = \boldsymbol{n}\cdot\boldsymbol{\sigma} \tag{48}$$

から

$$\boldsymbol{\Gamma}^2 = |\boldsymbol{n}|^2\boldsymbol{I} = \boldsymbol{I} \qquad (|\boldsymbol{n}| = 1) \tag{49}$$

また

$$\boldsymbol{h}\boldsymbol{\Gamma} = \boldsymbol{R}\cdot\boldsymbol{n} + \mathrm{i}(\boldsymbol{R}\times\boldsymbol{h})\cdot\boldsymbol{\sigma} \tag{50}$$
$$\boldsymbol{\Gamma}\boldsymbol{h} = \boldsymbol{n}\cdot\boldsymbol{R} + \mathrm{i}(\boldsymbol{h}\times\boldsymbol{R})\cdot\boldsymbol{\sigma} \tag{51}$$

(50) と (51) を (23) に代入すると

$$2\boldsymbol{R}\cdot\boldsymbol{n} = 0 \tag{52}$$

となる。

09/27/2011

(52) と (43) から

$$\boldsymbol{X}\delta k_x + \boldsymbol{Y}\delta k_y \perp \boldsymbol{n} \Longrightarrow \boldsymbol{X},\boldsymbol{Y}\perp\boldsymbol{n} \tag{53}$$

が言える。当然

$$\boldsymbol{X}\times\boldsymbol{Y}\perp\boldsymbol{X},\boldsymbol{Y} \tag{54}$$

であるので

$$\boldsymbol{X}\times\boldsymbol{Y} = \chi|\boldsymbol{X}\times\boldsymbol{Y}|\boldsymbol{n} \tag{55}$$
$$\chi = \pm 1 \qquad \text{(chirality of the Dirac Cone)} \tag{56}$$

と書ける。(38) と (43) から

$$h^2 = |\boldsymbol{X}\delta k_x + \boldsymbol{Y}\delta k_y|^2\sigma_0 \tag{57}$$
$$= \boldsymbol{X}\cdot\boldsymbol{X}\delta k_x^2 + \boldsymbol{Y}\cdot\boldsymbol{Y}\delta k_y^2 + 2(\boldsymbol{X}\cdot\boldsymbol{Y})\delta k_x\delta k_y \tag{58}$$
$$= \begin{pmatrix}\delta k_x & \delta k_y\end{pmatrix}\begin{pmatrix}\boldsymbol{X}\cdot\boldsymbol{X} & \boldsymbol{X}\cdot\boldsymbol{Y}\\ \boldsymbol{X}\cdot\boldsymbol{Y} & \boldsymbol{Y}\cdot\boldsymbol{Y}\end{pmatrix}\begin{pmatrix}\delta k_x\\ \delta k_y\end{pmatrix} \tag{59}$$

なお

$$\det\begin{pmatrix}\boldsymbol{X}\cdot\boldsymbol{X} & \boldsymbol{X}\cdot\boldsymbol{Y}\\ \boldsymbol{X}\cdot\boldsymbol{Y} & \boldsymbol{Y}\cdot\boldsymbol{Y}\end{pmatrix} = |\boldsymbol{X}|^2|\boldsymbol{Y}|^2 - |\boldsymbol{X}\cdot\boldsymbol{Y}|^2 \tag{60}$$
$$= |\boldsymbol{X}|^2|\boldsymbol{Y}|^2 - |\boldsymbol{X}|^2|\boldsymbol{Y}|^2\cos^2\theta \tag{61}$$
$$= |\boldsymbol{X}|^2|\boldsymbol{Y}|^2\sin^2\theta \tag{62}$$
$$= (|\boldsymbol{X}||\boldsymbol{Y}|\sin\theta)^2 \tag{63}$$
$$= |\boldsymbol{X}\times\boldsymbol{Y}|^2 \tag{64}$$
$$= (c\hbar)^4 \tag{65}$$

ここで

$$|\boldsymbol{X}\times\boldsymbol{Y}| \overset{\mathrm{def}}{=} (c\hbar)^2 \tag{66}$$

であり

$$c = \frac{\sqrt{|\boldsymbol{X}\times\boldsymbol{Y}|}}{\hbar} \tag{67}$$

は"光速"を表す。

$$\mathit{\Xi} \overset{\mathrm{def}}{=} \frac{1}{(c\hbar)^2}\begin{pmatrix}\boldsymbol{X}\cdot\boldsymbol{X} & \boldsymbol{X}\cdot\boldsymbol{Y}\\ \boldsymbol{X}\cdot\boldsymbol{Y} & \boldsymbol{Y}\cdot\boldsymbol{Y}\end{pmatrix} \tag{68}$$

は実対称行列なので対角化できる。$\mathit{\Xi}$ を対角化する行列を $\boldsymbol{V}$ とすると

$$\mathit{\Xi} = \boldsymbol{V}^\dagger\begin{pmatrix}\xi_1 & 0\\ 0 & \xi_2\end{pmatrix}\boldsymbol{V} \tag{69}$$

である。$\boldsymbol{V}$ はもちろん

$$\boldsymbol{V}^\dagger\boldsymbol{V} = \sigma_0 \tag{70}$$
$$\det\boldsymbol{V}^\dagger\boldsymbol{V} = 1 \tag{71}$$

を満たす。$\mathit{\Xi}$ は

$$\det\mathit{\Xi} = \xi_1\xi_2 = 1 \qquad \xi_1 > 0, \xi_2 > 0 \tag{72}$$
$$\mathrm{tr}\mathit{\Xi} > 0 \tag{73}$$

Ξ を用いると (59) は

$$h^2 = \begin{pmatrix}\delta k_x & \delta k_y\end{pmatrix} \Xi (c\hbar)^2 \begin{pmatrix}\delta k_x \\ \delta k_y\end{pmatrix} \tag{74}$$

$$= (c\hbar)^2 \begin{pmatrix}\delta k_x & \delta k_y\end{pmatrix} \boldsymbol{V}^\dagger \begin{pmatrix}\xi_1 & 0 \\ 0 & \xi_2\end{pmatrix} \boldsymbol{V} \begin{pmatrix}\delta k_x \\ \delta k_y\end{pmatrix} \tag{75}$$

さらに

$$\begin{pmatrix}K_1 \\ K_2\end{pmatrix} \stackrel{\text{def}}{=} \boldsymbol{V} \begin{pmatrix}\delta k_x \\ \delta k_y\end{pmatrix} \tag{76}$$

とすれば h^2 は

$$h^2 = (c\hbar)^2 \begin{pmatrix}K_1 & K_2\end{pmatrix} \begin{pmatrix}\xi_1 & 0 \\ 0 & \xi_2\end{pmatrix} \begin{pmatrix}K_1 \\ K_2\end{pmatrix} \tag{77}$$

$$= (c\hbar)^2 \left(\xi_1 K_1^2 + \xi_2 K_2^2\right) \tag{78}$$

$$= c^2 \left(\xi_1 p_1^2 + \xi_2 p_2^2\right) \tag{79}$$

となる。ただし

$$p_1 \stackrel{\text{def}}{=} \hbar K_1 \tag{80}$$

$$p_2 \stackrel{\text{def}}{=} \hbar K_2 \tag{81}$$

したがって

$$h\psi = E\psi \tag{82}$$

に対し、固有値 E は

$$E^2 = c^2 \left(\xi_1 p_1^2 + \xi_2 p_2^2\right) \tag{83}$$

から

$$E = \pm c \sqrt{\xi_1 p_1^2 + \xi_2 p_2^2} \tag{84}$$

$$= \pm c\bar{p} \qquad \text{(Dirac Cone)} \tag{85}$$

となる。$\bar{p}$ は averaged momentum、c は effective light velocity を表し $c \sim c_{\text{light}}/300$ である。

10/04/2011

Let us consider the Hamiltonian

$$H = \frac{\boldsymbol{p}^2}{2m} \tag{86}$$

How to get the Lorentz force?

$$\boldsymbol{p} \to \boldsymbol{p} - e\boldsymbol{A} = \boldsymbol{\Pi} \tag{87}$$

$$\boldsymbol{B} = \mathrm{rot}\boldsymbol{A} \tag{88}$$

$$B_z = \partial_x A_y - \partial_y A_x \qquad \boldsymbol{A}\text{ : vector potential} \tag{89}$$

Quantization

$$\boldsymbol{p} = -\mathrm{i}\hbar\nabla \tag{90}$$

$$p_x = -\mathrm{i}\hbar\partial_x \sim \hbar k_x \tag{91}$$

$$p_y = -\mathrm{i}\hbar\partial_y \sim \hbar k_y \tag{92}$$

Considering the inverse of the quantization

$$\delta k_x \approx \frac{1}{\hbar} p_x \to \Pi_x \tag{93}$$

$$\delta k_y \approx \frac{1}{\hbar} p_y \to \Pi_y \tag{94}$$

we obtain

$$\Pi_x = -\mathrm{i}\hbar\partial - eA_x(x, y) \tag{95}$$

$$\Pi_y = -\mathrm{i}\hbar\partial - eA_y(x, y) \tag{96}$$

$\boldsymbol{h}$ is rewritten by using Π_x, Π_y.

$$\boldsymbol{h} = \hbar^{-1} \left(\boldsymbol{X}\Pi_x + \boldsymbol{Y}\Pi_y\right) \tag{97}$$

their commutation relation is

$$[\Pi_x, \Pi_y] = [-\mathrm{i}\hbar\partial - eA_x, -\mathrm{i}\hbar\partial - eA_y] \tag{98}$$

$$= [-eA_x, -\mathrm{i}\hbar\partial_y] + [-\mathrm{i}\hbar\partial_x, -eA_y] \tag{99}$$

$$= \mathrm{i}\hbar e \left\{[A_x, \partial_y] + [\partial_x, A_y]\right\} \tag{100}$$

$$= \mathrm{i}\hbar e \left\{A_x\partial_y - \partial_y A_x + \partial_x A_y - \partial_x A_y\right\} \tag{101}$$

$$= \mathrm{i}\hbar e \left(\partial_x A_y - \partial_y A_x\right) \tag{102}$$

$$= \mathrm{i}\hbar e B \qquad (\boldsymbol{B} = \mathrm{rot}\boldsymbol{A}) \tag{103}$$

We assume $B < 0$, because electron charge e is negative.

$$\boldsymbol{h}^2 = \hbar^2[(\boldsymbol{X}\Pi_x + \boldsymbol{Y}\Pi_y)\cdot\boldsymbol{\sigma}]^2 \tag{104}$$
$$= \hbar^2\left\{(\boldsymbol{X}\Pi_x + \boldsymbol{Y}\Pi_y)\cdot(\boldsymbol{X}\Pi_x + \boldsymbol{Y}\Pi_y)\sigma_0 + \mathrm{i}(\boldsymbol{X}\Pi_x + \boldsymbol{Y}\Pi_y)\times(\boldsymbol{X}\Pi_x + \boldsymbol{Y}\Pi_y)\boldsymbol{\sigma}\right\} \tag{105}$$
$$= \hbar^2\left\{\left(\boldsymbol{X}\cdot\boldsymbol{X}\Pi_x^2 + \boldsymbol{X}\cdot\boldsymbol{Y}\Pi_x\Pi_y + \boldsymbol{Y}\cdot\boldsymbol{X}\Pi_y\Pi_x + \boldsymbol{Y}\cdot\boldsymbol{Y}\Pi_y^2\right)\sigma_0 + \mathrm{i}\left(\boldsymbol{X}\times\boldsymbol{Y}\Pi_x\Pi_y + \boldsymbol{Y}\times\boldsymbol{X}\Pi_y\Pi_x\right)\cdot\boldsymbol{\sigma}\right\} \tag{106}$$
$$= \hbar^2\begin{pmatrix}\Pi_x & \Pi_y\end{pmatrix}\begin{pmatrix}\boldsymbol{X}\cdot\boldsymbol{X} & \boldsymbol{X}\cdot\boldsymbol{Y}\\ \boldsymbol{X}\cdot\boldsymbol{Y} & \boldsymbol{Y}\cdot\boldsymbol{Y}\end{pmatrix}\begin{pmatrix}\Pi_x\\ \Pi_y\end{pmatrix}\sigma_0 + \mathrm{i}\,\underbrace{[\Pi_x,\Pi_y]}_{\mathrm{i}\hbar eB}\underbrace{(\boldsymbol{X}\times\boldsymbol{Y})}_{\chi(c\hbar)^2}\cdot\boldsymbol{\sigma} \tag{107}$$
$$= c^2\begin{pmatrix}\Pi_x & \Pi_y\end{pmatrix}\Xi\begin{pmatrix}\Pi_x\\ \Pi_y\end{pmatrix}\sigma_0 - c^2\hbar eB\gamma \tag{108}$$
$$= c^2(\xi_1^2\Pi_1^2 + \xi_2^2\Pi_2^2)\sigma_0 - c^2\hbar eB\gamma \tag{109}$$

where

$$\gamma \overset{\text{def}}{=} \boldsymbol{n}_\gamma\cdot\boldsymbol{\sigma} \qquad (\gamma = \pm 1) \tag{110}$$
$$\begin{pmatrix}\Pi_1\\ \Pi_2\end{pmatrix} \overset{\text{def}}{=} \boldsymbol{V}\begin{pmatrix}\Pi_x\\ \Pi_y\end{pmatrix} = \begin{pmatrix}V_{1x}\Pi_x + V_{1y}\Pi_y\\ V_{2x}\Pi_x + V_{2y}\Pi_y\end{pmatrix} \tag{111}$$

Now we difine

$$\Pi_1' \overset{\text{def}}{=} \xi_1\Pi_1 \tag{112}$$
$$\Pi_2' \overset{\text{def}}{=} \xi_2\Pi_1 \tag{113}$$

satisfying

$$[\Pi_1',\Pi_2'] = \underbrace{\xi_1\xi_2}_{\det\Xi=1}[\Pi_1,\Pi_2] = [\Pi_1,\Pi_2] = [\Pi_1,\Pi_2] = [\Pi_x,\Pi_y] \tag{114}$$

guaranteed by

$$[\Pi_1,\Pi_2] = [V_{1x}\Pi_x + V_{1y}\Pi_y, V_{2x}\Pi_x + V_{2y}\Pi_y] \tag{115}$$
$$= V_{1x}V_{2y}[\Pi_x,\Pi_y] + V_{1y}V_{2x}[\Pi_y,\Pi_x] \tag{116}$$
$$= \underbrace{\left(V_{1x}V_{2y} - V_{1y}V_{2x}\right)}_{\det\boldsymbol{V}=1}[\Pi_x,\Pi_y] \tag{117}$$
$$= [\Pi_x,\Pi_y] \tag{118}$$

which is obvious because $\boldsymbol{V}$ is orthogonal.

$$\boldsymbol{h}^2 = c^2\boldsymbol{\Pi}'^2\sigma_0 - c^2\hbar eB\chi\gamma \tag{119}$$

Using magnetic length

$$l \overset{\text{def}}{=} \sqrt{\frac{\hbar}{eB}} \tag{120}$$

the commutation relation (114) is rewritten as

$$[\Pi_1',\Pi_2'] = \mathrm{i}\hbar eB = \mathrm{i}\hbar\frac{\hbar}{l^2} = \mathrm{i}\left(\frac{\hbar}{l}\right)^2 \tag{121}$$

so,

$$[\frac{l}{\hbar}\Pi_x', \frac{l}{\hbar}\Pi_y'] = \mathrm{i} \tag{122}$$

Let us introduce an operator

$$a \overset{\text{def}}{=} \frac{1}{\sqrt{2}}\frac{l}{\hbar}\left(\Pi_1' + \mathrm{i}\Pi_2'\right) \tag{123}$$

satisfying

$$[a,a^\dagger] = \frac{1}{2}\left(\frac{l}{\hbar}\right)^2[\Pi_1' + \mathrm{i}\Pi_2', \Pi_1' - \mathrm{i}\Pi_2'] \tag{124}$$
$$= \frac{1}{2}\left(\frac{l}{\hbar}\right)^2(-2)\left(\frac{\hbar}{l}\right)^2 \tag{125}$$
$$= 1 \tag{126}$$

and

$$a^\dagger a = \frac{1}{2}\left(\Pi_1' - \mathrm{i}\Pi_2'\right)\left(\Pi_1' + \mathrm{i}\Pi_2'\right) \tag{127}$$
$$= \frac{1}{2}\left(\frac{l}{\hbar}\right)^2\left(\Pi_1'^2 + \mathrm{i}\Pi_2'^2 + \mathrm{i}\underbrace{[\Pi_1',\Pi_2']}_{\mathrm{i}\hbar eB}\right) \tag{128}$$
$$= \frac{1}{2}\left(\frac{l}{\hbar}\right)^2\left(\boldsymbol{\Pi}'^2 - \mathrm{i}\hbar eB\right) \tag{129}$$
$$= \frac{l^2}{2\hbar^2}\boldsymbol{\Pi}'^2 - \underbrace{\frac{l^2}{2\hbar^2}\mathrm{i}\hbar eB}_{-1/2} \tag{130}$$
$$\Longleftrightarrow \quad a^\dagger a + \frac{1}{2} = \frac{l^2}{2\hbar^2}\boldsymbol{\Pi}'^2 \tag{131}$$

Using (131) and

$$n \overset{\text{def}}{=} a^\dagger a \tag{132}$$
$$\hbar\omega \overset{\text{def}}{=} \frac{eB}{m} = 2c^2eB \tag{133}$$

the following relationship is obtained:

$$\hbar\omega\left(n + \frac{1}{2}\right) = \hbar\omega\frac{l^2}{2\hbar^2}\boldsymbol{\Pi}'^2 \tag{134}$$
$$= \left(\frac{\hbar}{l}\right)^2\frac{1}{2m}\left(\frac{l}{\hbar}\right)^2\boldsymbol{\Pi}'^2 \tag{135}$$
$$= \frac{\boldsymbol{\Pi}'^2}{2m} = c^2\boldsymbol{\Pi}'^2 \tag{136}$$

So $\boldsymbol{h}^2$ can be expressed as

$$\boldsymbol{h}^2 = \hbar\omega\left(n + \frac{1}{2}\right)\sigma_0 - \frac{1}{2m}\hbar eB\chi\gamma \tag{137}$$
$$= \hbar\omega\left(n + \frac{1}{2}\right) - \frac{1}{2}\hbar\omega\chi\gamma \tag{138}$$

If $\gamma = \chi = \pm 1$, the eigenvalue becomes

$$E = \pm c\sqrt{2\hbar eB|n|} \qquad \text{(LL of Dirac fermion)} \tag{139}$$

10/11/2011

$$\begin{aligned}
[\Pi_i, \Pi_j] &= [-\mathrm{i}\hbar\partial_i - eA_i, -\mathrm{i}\hbar\partial_j - eA_j] && (140)\\
&= [-\mathrm{i}\hbar\partial_i, -eA_j] + [-eA_i, -\mathrm{i}\hbar\partial_j] && (141)\\
&= (-\mathrm{i}\hbar)(-e)\left\{[\partial_i, A_j] + [A_i, \partial_j]\right\} && (142)\\
&= \mathrm{i}\hbar e(\partial_i A_j - \partial_j A_i) && (143)\\
&= \mathrm{i}\hbar e \epsilon_{ijk}(\mathrm{rot}\boldsymbol{A})_k && (144)\\
&= \mathrm{i}\hbar e \epsilon_{ijk} B_k && (145)
\end{aligned}$$

何故ならば

$$\begin{aligned}
\epsilon_{ijk}B_k &= \epsilon_{ijk}\epsilon_{abk}\partial_a A_b && (146)\\
&= (\delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja})\partial_a A_b && (147)\\
&= \partial_i A_j - \partial_j A_i && (148)
\end{aligned}$$

Let us introduce

$$\begin{aligned}
\Pi_\perp &= \boldsymbol{n} \times \boldsymbol{\Pi} && (149)\\
\Pi_{\perp i} &= \epsilon_{ijk} n_j \Pi_k && (150)
\end{aligned}$$

whose commutation relation is

$$\begin{aligned}
[\Pi_{\perp i}, \Pi_{\perp j}] &= [\epsilon_{iab} n_a \Pi_b, \epsilon_{jcd} n_c \Pi_d] && (151)\\
&= \epsilon_{iab}\epsilon_{jcd} n_a n_c [\Pi_b, \Pi_d] && (152)\\
&= \mathrm{i}\hbar eB \epsilon_{iab}\epsilon_{jcd}\epsilon_{bde} n_a n_c n_e && (153)\\
&= \mathrm{i}\hbar eB \epsilon_{jcd}(\delta_{jd}\delta_{ae} - \delta_{ie}\delta_{ad}) n_a n_c n_e && (154)\\
&= \mathrm{i}\hbar eB \left(\epsilon_{jci} n_a n_c n_a - \epsilon_{jca} n_d n_c n_i\right) && (155)\\
&= \mathrm{i}\hbar eB \epsilon_{ijc} n_c |\boldsymbol{n}|^2 && (156)\\
&= \mathrm{i}\hbar eB \epsilon_{ijk} n_k && (157)
\end{aligned}$$

The guiding center $\boldsymbol{R}$ is defined as follows:

$$\begin{aligned}
\boldsymbol{R} &\overset{\text{def}}{=} \boldsymbol{r} - \frac{l^2}{\hbar}\boldsymbol{\Pi}_\perp && (158)\\
R_i &= r_i - \frac{l^2}{\hbar}\Pi_{\perp i} && (159)
\end{aligned}$$

$$\begin{aligned}
[r_i, \Pi_{\perp j}] &= [r_i, \epsilon_{jab} n_a \Pi_b] && (160)\\
&= \epsilon_{jab} n_a \underbrace{[r_i, \Pi_b]}_{=[r_i, p_b]=\delta_{ib}\mathrm{i}\hbar} && (161)\\
&= \epsilon_{jai} n_a \mathrm{i}\hbar && (162)\\
&= \mathrm{i}\hbar\epsilon_{ija} n_a && (163)
\end{aligned}$$

$$\begin{aligned}
[R_i, \Pi_{\perp j}] &= [r_i, \Pi_{\perp j}] - \frac{l^2}{\hbar}[\Pi_{\perp i}, \Pi_{\perp j}] && (164)\\
&= \mathrm{i}\hbar\epsilon_{ija} n_a - \frac{l^2}{\hbar}\mathrm{i}\left(\frac{\hbar}{l}\right)^2 \epsilon_{ija} n_a && (165)\\
&= 0 \qquad \text{(conserved)} && (166)
\end{aligned}$$

$$\begin{aligned}
[R_i, R_j] &= [r_i - \frac{l^2}{\hbar}\Pi_{\perp i}, r_j - \frac{l^2}{\hbar}\Pi_{\perp j}] && (167)\\
&= -\frac{l^2}{\hbar}\left\{[\Pi_{\perp i}, r_j] + [r_i, \Pi_{\perp j}] + \left(\frac{l}{\hbar}\right)^2 [\Pi_{\perp i}, \Pi_{\perp j}]\right\} && (168)\\
&= -\frac{l^2}{\hbar}\left\{-[r_j, \Pi_{\perp i}] + \mathrm{i}\hbar\epsilon_{ija} n_a\right\} + \left(\frac{l^2}{\hbar}\right)^2 \mathrm{i}\left(\frac{\hbar}{l}\right)^2 \epsilon_{ija} n_a && (169)\\
&= -\mathrm{i}2l^2\epsilon_{ija} n_a + \mathrm{i}l^2\epsilon_{ija} n_a && (170)\\
&= -\mathrm{i}l^2\epsilon_{ija} n_a && (171)
\end{aligned}$$

10/18/2011 以降

$$\text{Hall conductance}: \sigma_{xy} = \frac{e^2}{\hbar}C \tag{172}$$

$$\text{Chen number}: C = \frac{1}{2\pi\mathrm{i}}\int_{\bigcirc} F \tag{173}$$

$$\begin{aligned}
A &= \langle\psi|\mathrm{d}\psi\rangle && (174)\\
&= \langle\psi|\frac{\partial}{\partial\phi_x}|\psi\rangle \mathrm{d}\phi_x + \langle\psi|\frac{\partial}{\partial\phi_y}|\psi\rangle \mathrm{d}\phi_y && (175)\\
&= \langle\psi|\partial_\mu\psi\rangle \mathrm{d}\phi_\mu && (176)
\end{aligned}$$

$$
\begin{aligned}
F &= \mathrm{d}A && (177)\\
&= \mathrm{d}\phi_\mu \partial_\mu A && (178)\\
&= \mathrm{d}\phi_x \partial_x A + \mathrm{d}\phi_x \partial_x A && (179)\\
&= \mathrm{d}\phi_x \partial_x (\mathrm{d}\phi_x \langle\psi|\partial_x\psi\rangle + \mathrm{d}\phi_y \langle\psi|\partial_y\psi\rangle) + \mathrm{d}\phi_y \partial_y (\mathrm{d}\phi_x \langle\psi|\partial_x\psi\rangle + \mathrm{d}\phi_y \langle\psi|\partial_y\psi\rangle) && (180)\\
&= \mathrm{d}\phi_x \mathrm{d}\phi_x \partial_x \langle\psi|\partial_x\psi\rangle + \mathrm{d}\phi_x \mathrm{d}\phi_y \partial_x \langle\psi|\partial_y\psi\rangle + \mathrm{d}\phi_y \mathrm{d}\phi_x \partial_y \langle\psi|\partial_x\psi\rangle + \mathrm{d}\phi_y \mathrm{d}\phi_y \partial_y \langle\psi|\partial_y\psi\rangle && (181)\\
&= \mathrm{d}\phi_x \mathrm{d}\phi_y (\partial_x \langle\psi|\partial_y\psi\rangle - \partial_y \langle\psi|\partial_x\psi\rangle) && (182)\\
&= \mathrm{d}\phi_x \mathrm{d}\phi_y ((\langle\partial_x\psi|\partial_y\psi\rangle + \langle\psi|\partial_x\partial_y\psi\rangle - \langle\partial_y\psi|\partial_x\psi\rangle - \langle\psi|\partial_y\partial_x\psi\rangle) && (183)\\
&= \mathrm{d}\phi_x \mathrm{d}\phi_y ((\langle\partial_x\psi|\partial_y\psi\rangle - \langle\partial_y\psi|\partial_x\psi\rangle) && (184)
\end{aligned}
$$

So,

$$
\begin{aligned}
C &= \frac{1}{2\pi \mathrm{i}} \int_0^{2\pi} \mathrm{d}\phi_x \int_0^{2\pi} \mathrm{d}\phi_y (\langle\partial_x\psi|\partial_y\psi\rangle - \langle\partial_y\psi|\partial_x\psi\rangle) && (185)\\
&= \frac{1}{2\pi \mathrm{i}} \int_0^{2\pi} \mathrm{d}\phi_x \int_0^{2\pi} \mathrm{d}\phi_y (\mathrm{rot}\boldsymbol{A})_z && (186)
\end{aligned}
$$

$$|\psi\rangle_g \stackrel{\text{def}}{=} |\psi\rangle g \qquad (187)$$

Using

$$\langle\psi|\psi\rangle = 1 \qquad (188)$$

the following equation is obtained:

$$
\begin{aligned}
{}_g\langle\psi|\psi\rangle_g &= g^*\langle\psi|\psi\rangle g && (189)\\
&= |g|^2 = 1 \Rightarrow g \in U(1) \quad (g = \mathrm{e}^{\mathrm{i}\theta}, \theta \in \mathbb{R}) && (190)
\end{aligned}
$$

$$
\begin{aligned}
A_x^g &= {}_g\langle\psi|\partial_x\psi\rangle_g && (191)\\
&= g^*\langle\psi|\partial_x(|\psi\rangle)g && (192)\\
&= g^*(\langle\psi|\partial_x\psi\rangle + \langle\psi|\psi\rangle\partial_x)g && (193)\\
&= g^* A_x g + g^*\partial_x g && (194)\\
&= A_x + \mathrm{e}^{-\mathrm{i}\theta}(\mathrm{i}\partial_x\theta)\mathrm{e}^{\mathrm{i}\theta} && (195)\\
&= A_x + \mathrm{i}\partial_x\theta \qquad \text{(gauge transformation)} && (196)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{rot}\boldsymbol{A}^g &= \nabla \times \boldsymbol{A}^g && (197)\\
&= \nabla \times \boldsymbol{A} + \mathrm{i}\nabla \times \nabla\theta && (198)\\
&= \mathrm{rot}\boldsymbol{A} && (199)
\end{aligned}
$$

So Chern number is gauge invariant.

$$\partial_x|\psi\rangle \stackrel{\text{def}}{=} \lim_{h\to 0} \frac{|\psi(\phi_x + h, \phi_y)\rangle - |\psi(\phi_x, \phi_y)\rangle}{h} \qquad (200)$$

is not well defined. One way to avoid the difficulty is gauge fixing.

$$P \stackrel{\text{def}}{=} |\psi\rangle\langle\psi| \qquad (201)$$

$$
\begin{aligned}
P^g &= |\psi\rangle_{g\,g}\langle\psi| && (202)\\
&= |\psi\rangle g g^* \langle\psi| && (203)\\
&= |\psi\rangle\langle\psi| = P && (204)
\end{aligned}
$$

take arbitary$|T\rangle$

$$
\begin{aligned}
P|T\rangle &= |\psi_u^T\rangle \qquad \text{(smooth)} && (205)\\
H|\psi_u^T\rangle &= E|\psi_u^T\rangle && (206)
\end{aligned}
$$

Needs normalization

$$
\begin{aligned}
N &= \langle\psi_n^T|\psi_n^T\rangle && (207)\\
&= \langle\psi|T\rangle^*\langle\psi|\psi\rangle\langle\psi|T\rangle && (208)\\
&= |\langle\psi|T\rangle|^2 && (209)
\end{aligned}
$$

$$|\psi^T\rangle = \frac{|\psi_u^T\rangle}{\sqrt{N}} \qquad \text{(gauge fix)} \qquad (210)$$