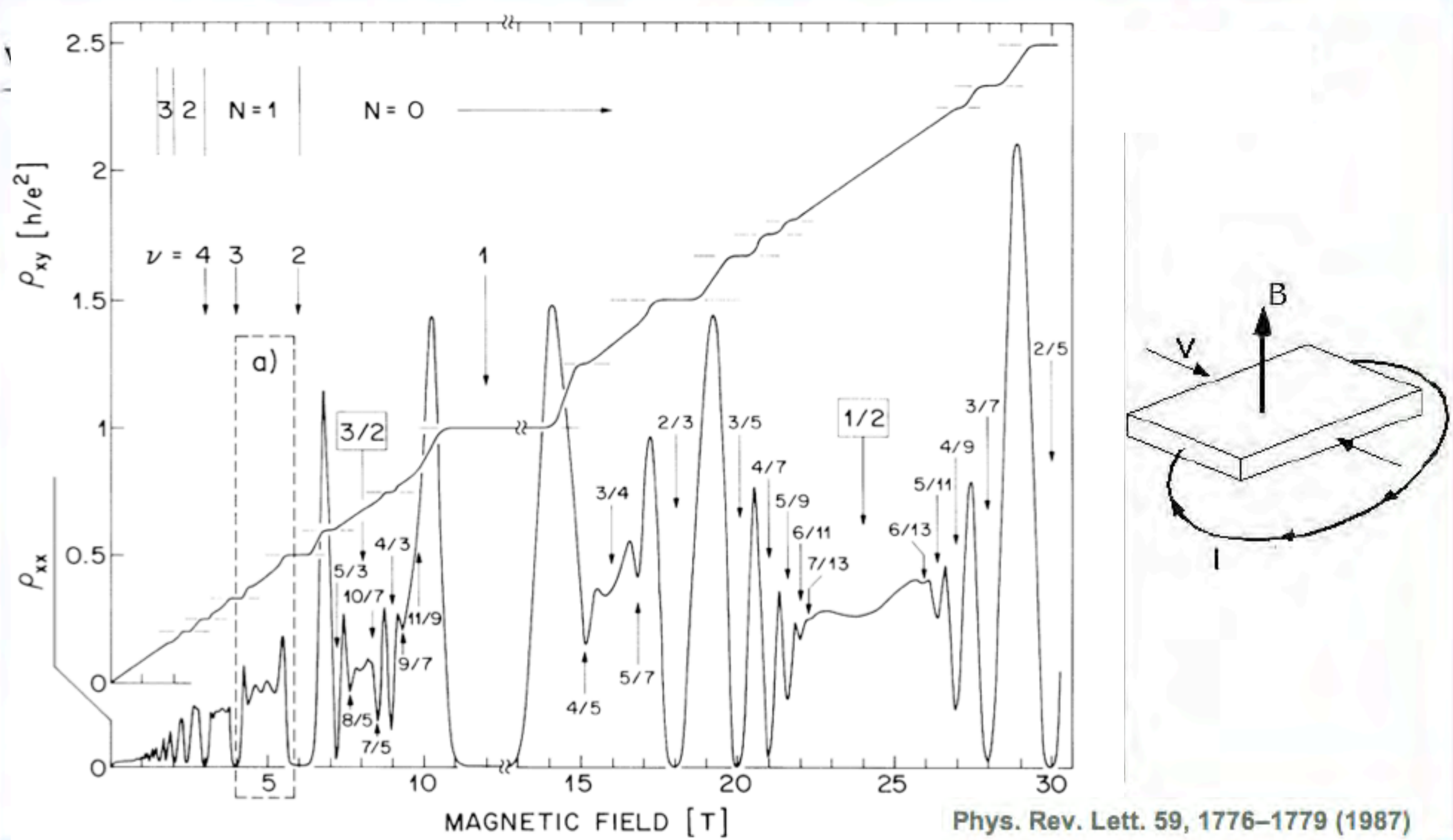
- Topological characterization by edge states
  - Quantization of Hall conductance (graphene as an example)
  - Laughlin argument & edge states
  - Topological number & edge states

Topol. char. by edges

Quantum Hall Effect '80, K.v.Klitzing et al.

Quantization of the Hall conductance σ_{xy} with anomalous accuracy: $I = \sigma_{xy} V$



Phys. Rev. Lett. 59, 1776–1779 (1987)

R. Willett et al.

Topol. char. by edges

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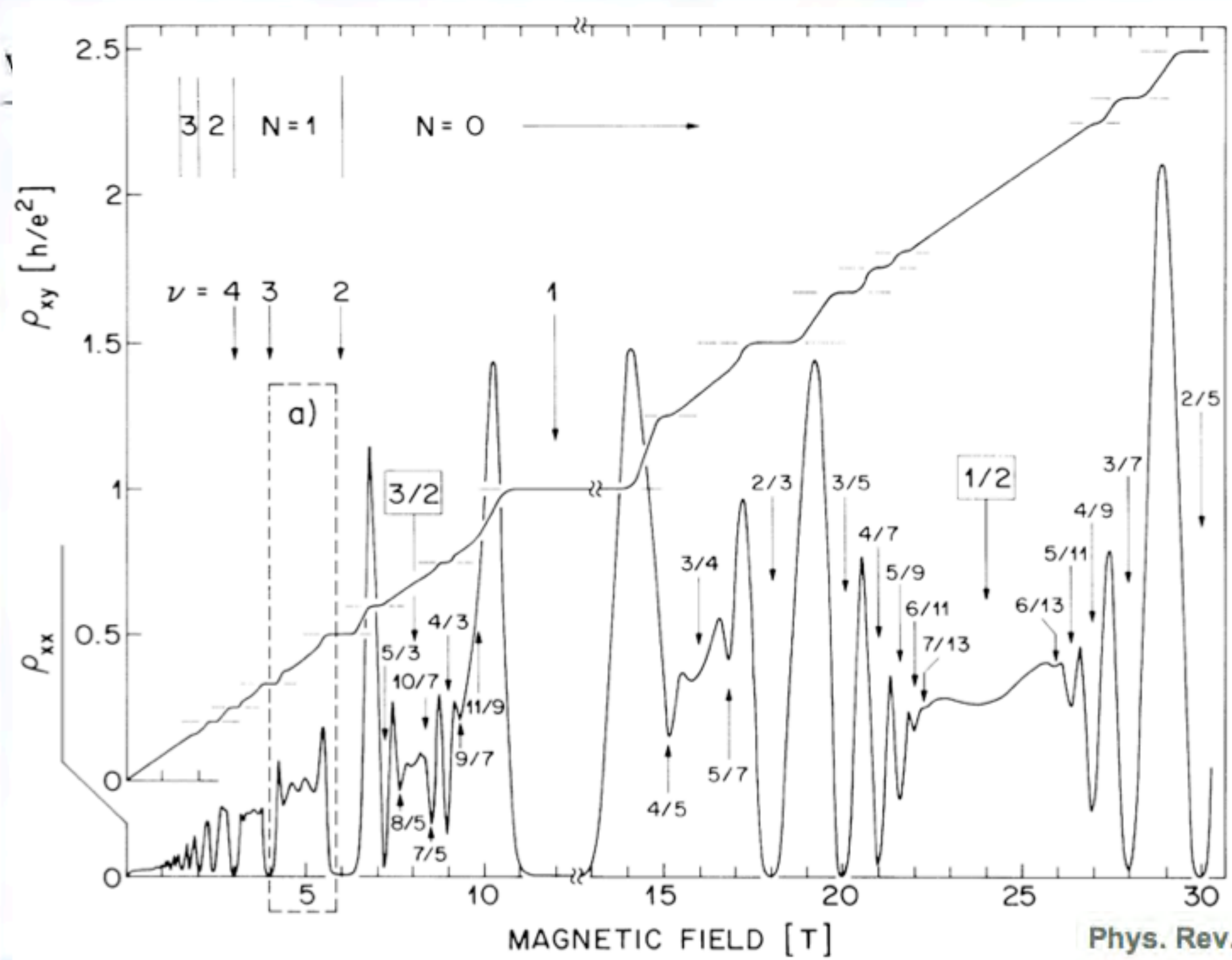
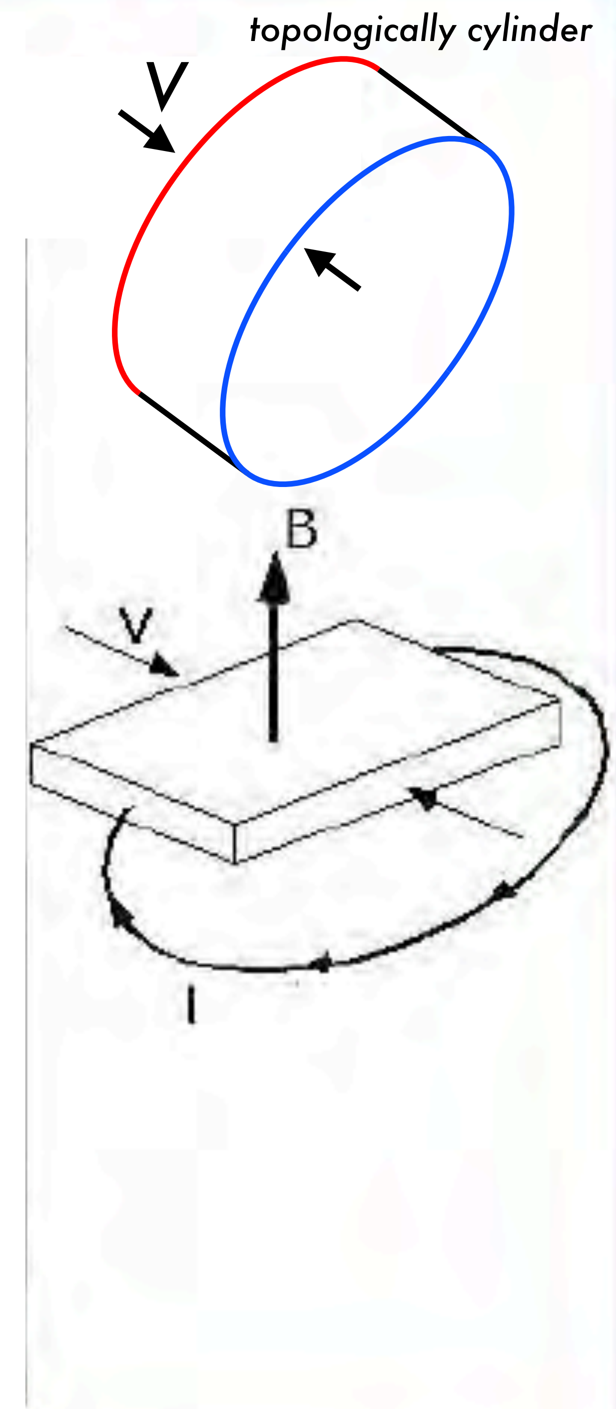




Phys. Rev. Lett. 59, 1776–1779 (1987)

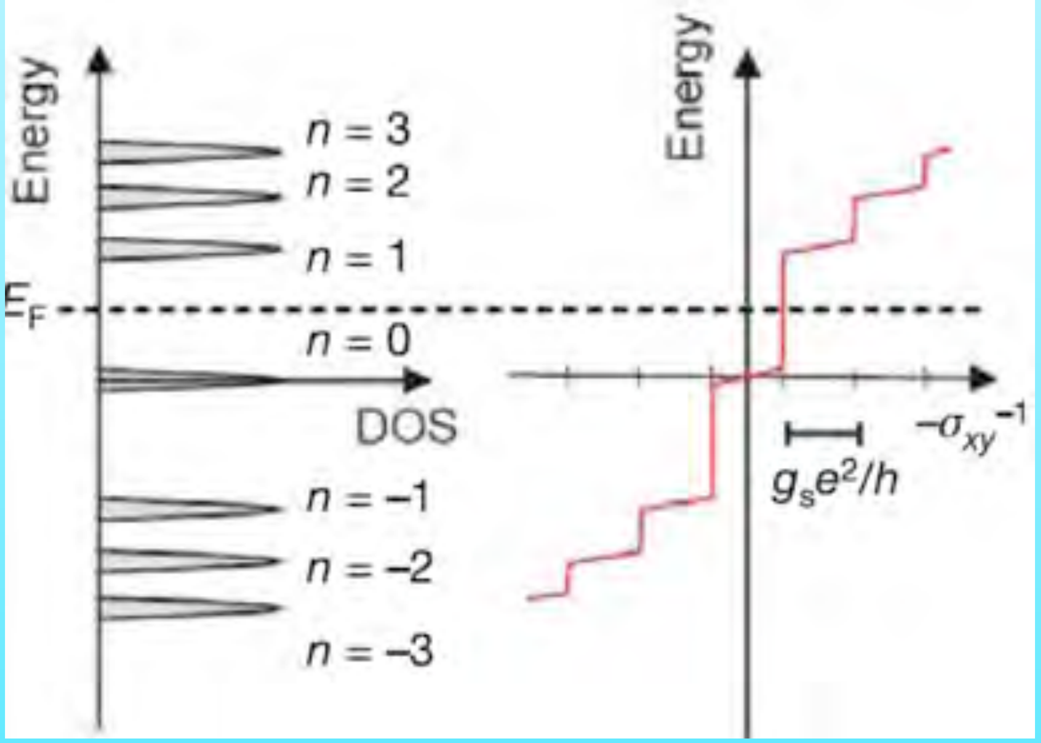
R. Willett et al.

Graphene

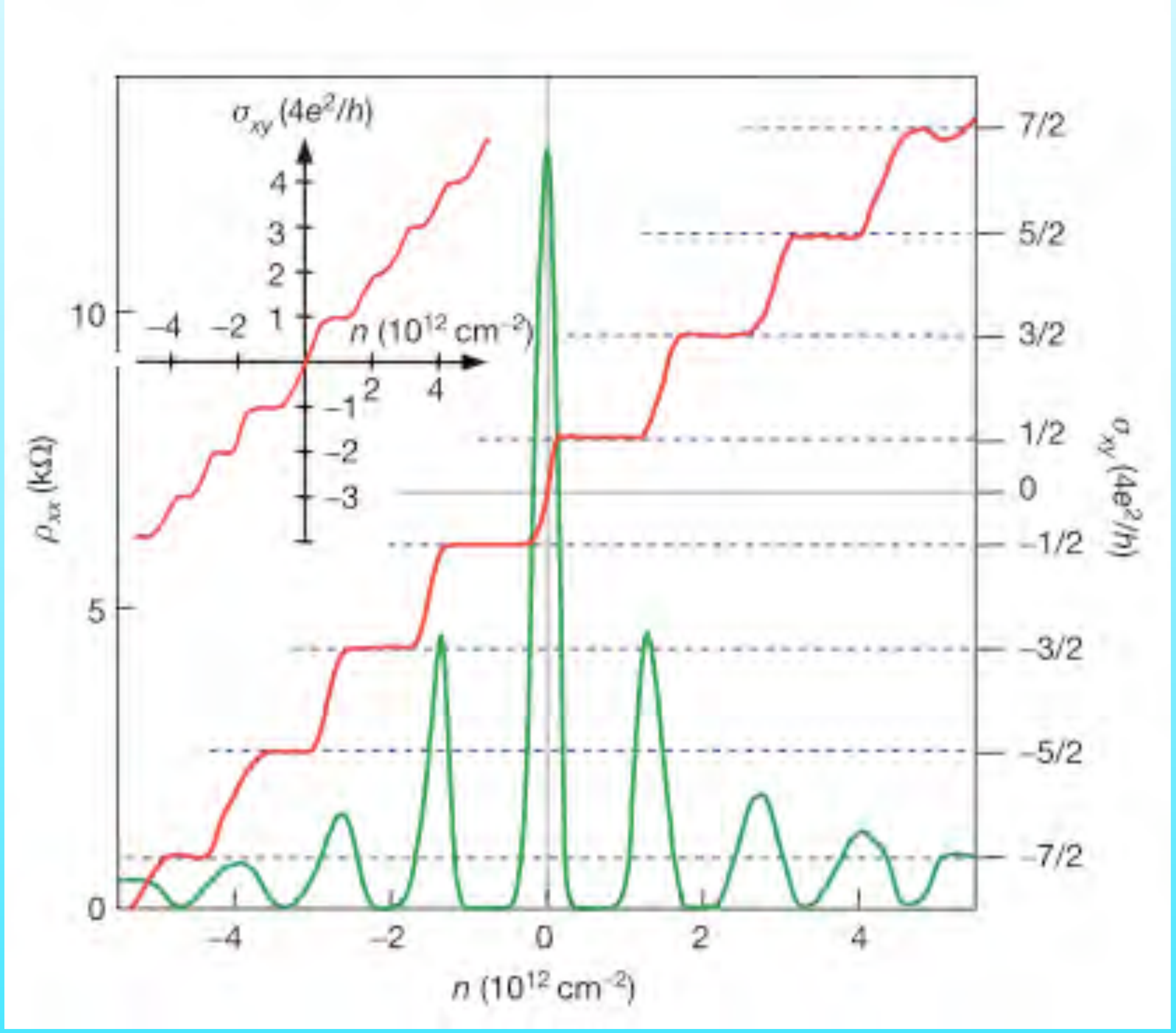
Topol. char. by edges

☆ *Anomalous QHE of gapless Dirac Fermions*

$$\sigma_{xy} = \frac{e^2}{h}(2n+1),\ \ n = 0, \pm 1, \pm 2, \cdots$$
$$= 2\frac{e^2}{h}(n + \frac{1}{2})$$



Zhang et al. Nature 2005



Novoselov et al. Nature 2005

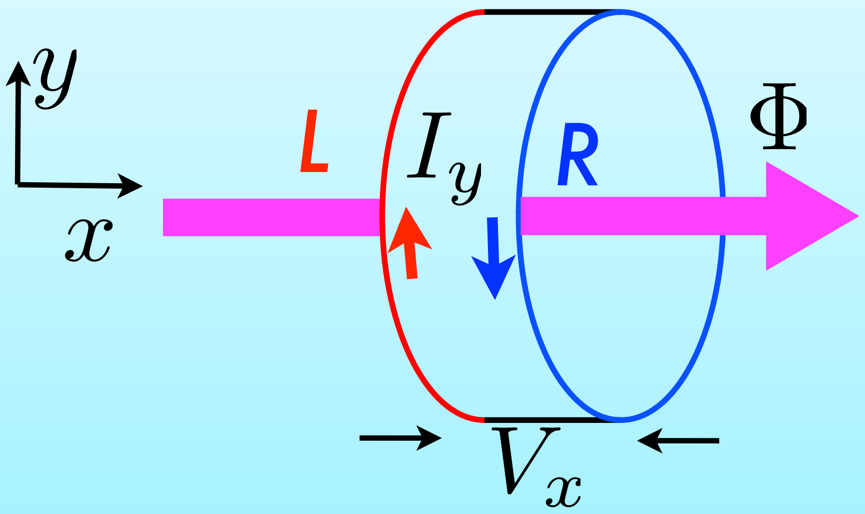
Topol. char. by edges

Stability of the quantized Hall Conductance

☆ Gauge invariance and quantization of σ_{xy}

Laughlin '81

adiabatic process to increase Φ



$$A \to A' = A + \nabla\Phi, \quad \delta\Phi = \int_{\bigcirc} (A' - A)$$

$$\psi \to \psi' = e^{i2\pi\delta\Phi/\Phi_0}\,\psi$$ one particle state

$$\delta\Phi = \Phi_0 = \frac{e}{h} \longrightarrow \psi' = \psi$$

flux quantum

Gauge transformation

Byers-Yang formula

$$I_y = \frac{\Delta E}{\Delta\Phi} = \frac{neV_x}{h/e} = \underbrace{\boxed{n\left(\frac{e^2}{h}\right)}}_{\sigma_{yx}} V_x = \sigma_{yx} V_x$$

$$\Delta\Phi = \Phi_0 = \frac{h}{e}, \quad \Delta E = n \cdot eV_x$$

All states are invariant up to phase after the process:

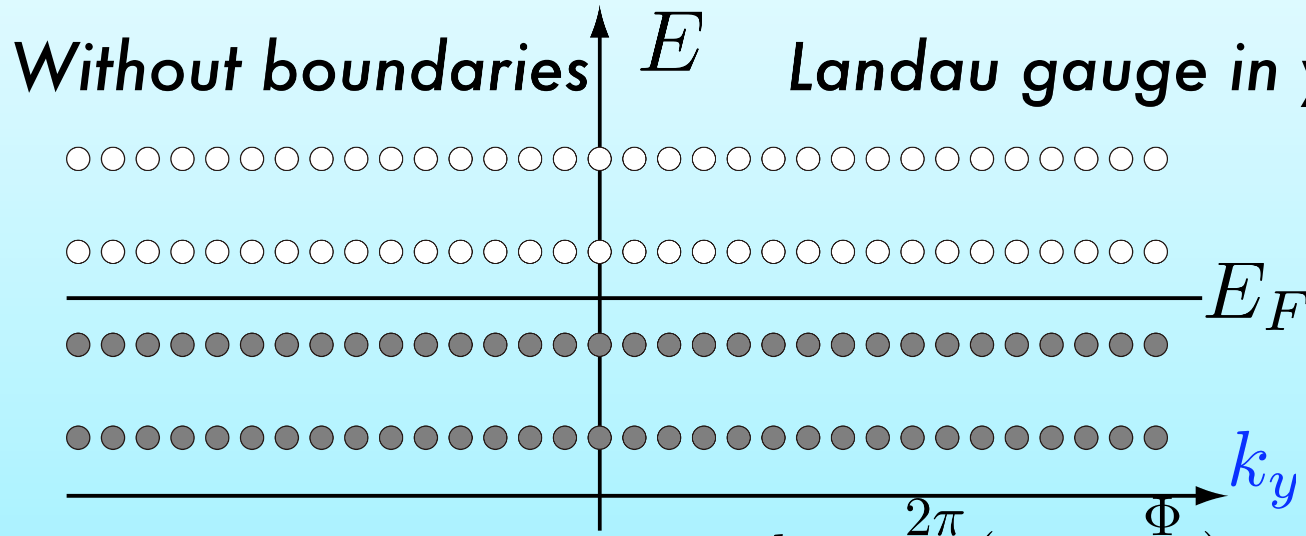
Some n states are carried from L to the R

n : generic integer (but undetermined)

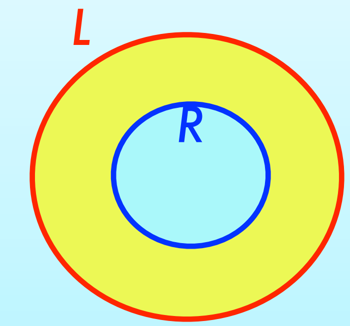
Topol. char. by edges

Stability of the quantized Hall Conductance

☆ Edge states and Hall conductance σ_{xy} Halperin '82



$$H_{2D} = \sum_{k_y} H_{1D}(k_y)$$

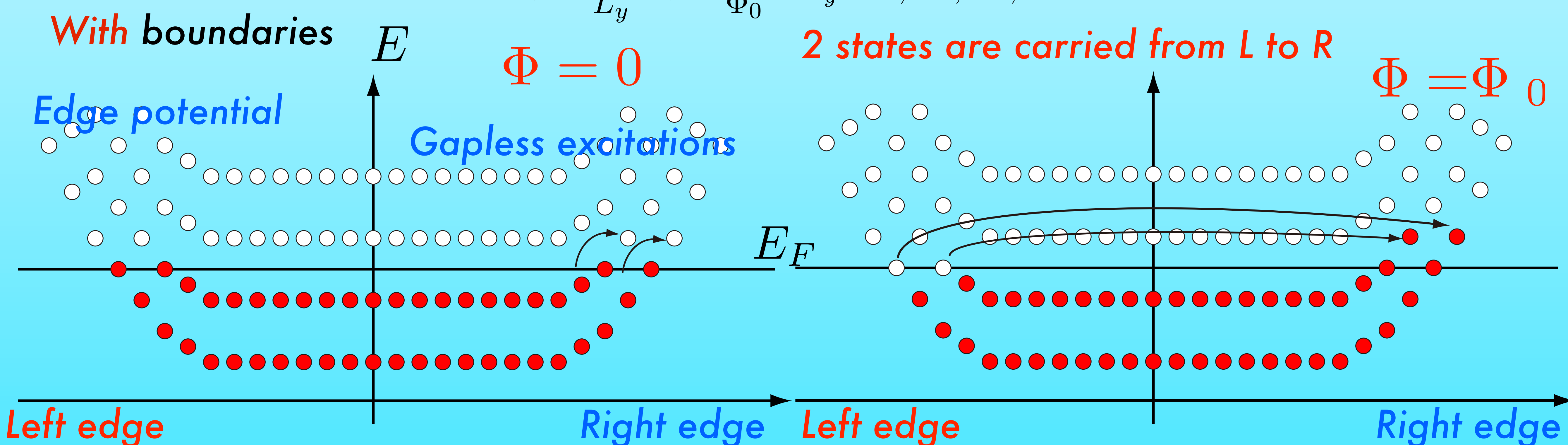


$H_{1D}(k_y)$: harmonic osc. centered at

$$\langle x \rangle \sim \ell_B^2 k_y$$

$$k_y = \frac{2\pi}{L_y}(n_y + \frac{\Phi}{\Phi_0}) \quad n_y = 0, \pm 1, \pm 2, \cdots$$

2 states are carried from L to R

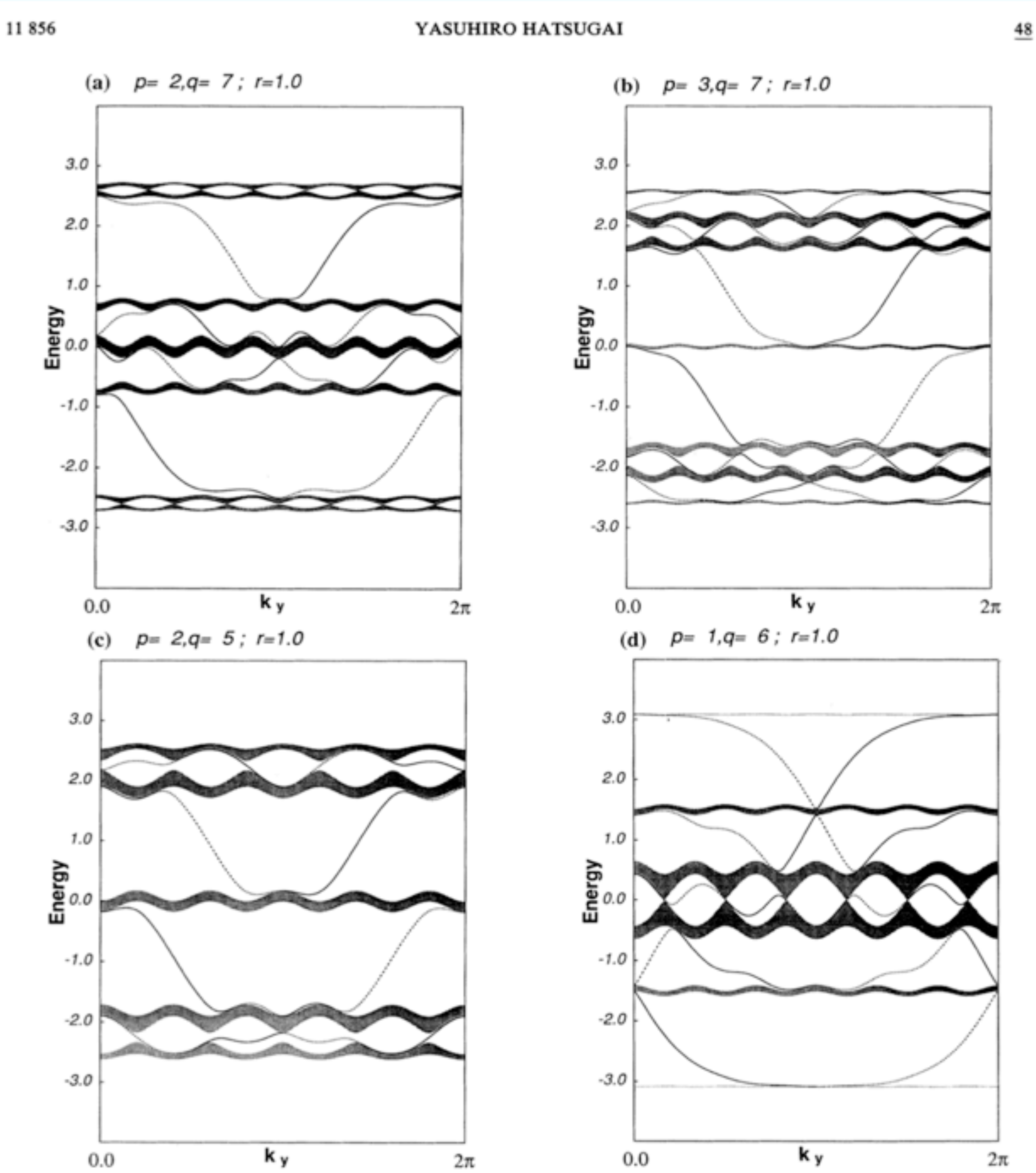


Laughlin's undetermined n : # of Landau Levels below E_F

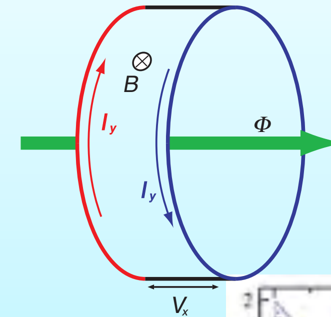
Edge states are essential in the QHE !

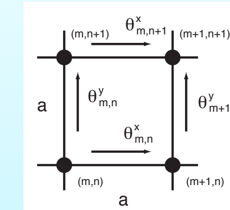
Quantum Hall edge states : Lattice electrons on cylinder

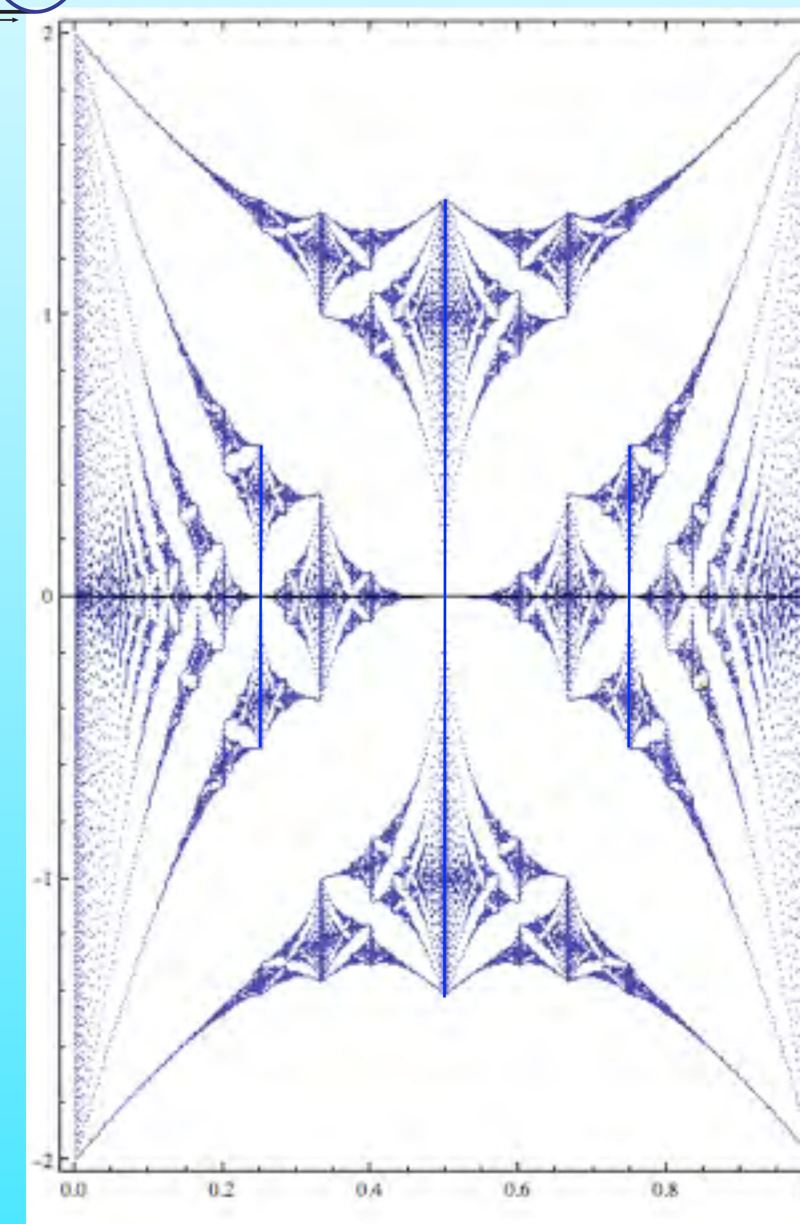
Topol. char. by edges



Y. H, Phys. Rev. B 48, 11851–11862 (1993)







C. Albrecht, J.H. Smet, K. von Klitzing, D. Weiss, V. Umansky, H. Schweizer: Evidence of Hofstadter's Fractal Energy Spectrum in the Quantized Hall Conductance. Phys. Rev. Lett. 86(1), 147-150 (2001).

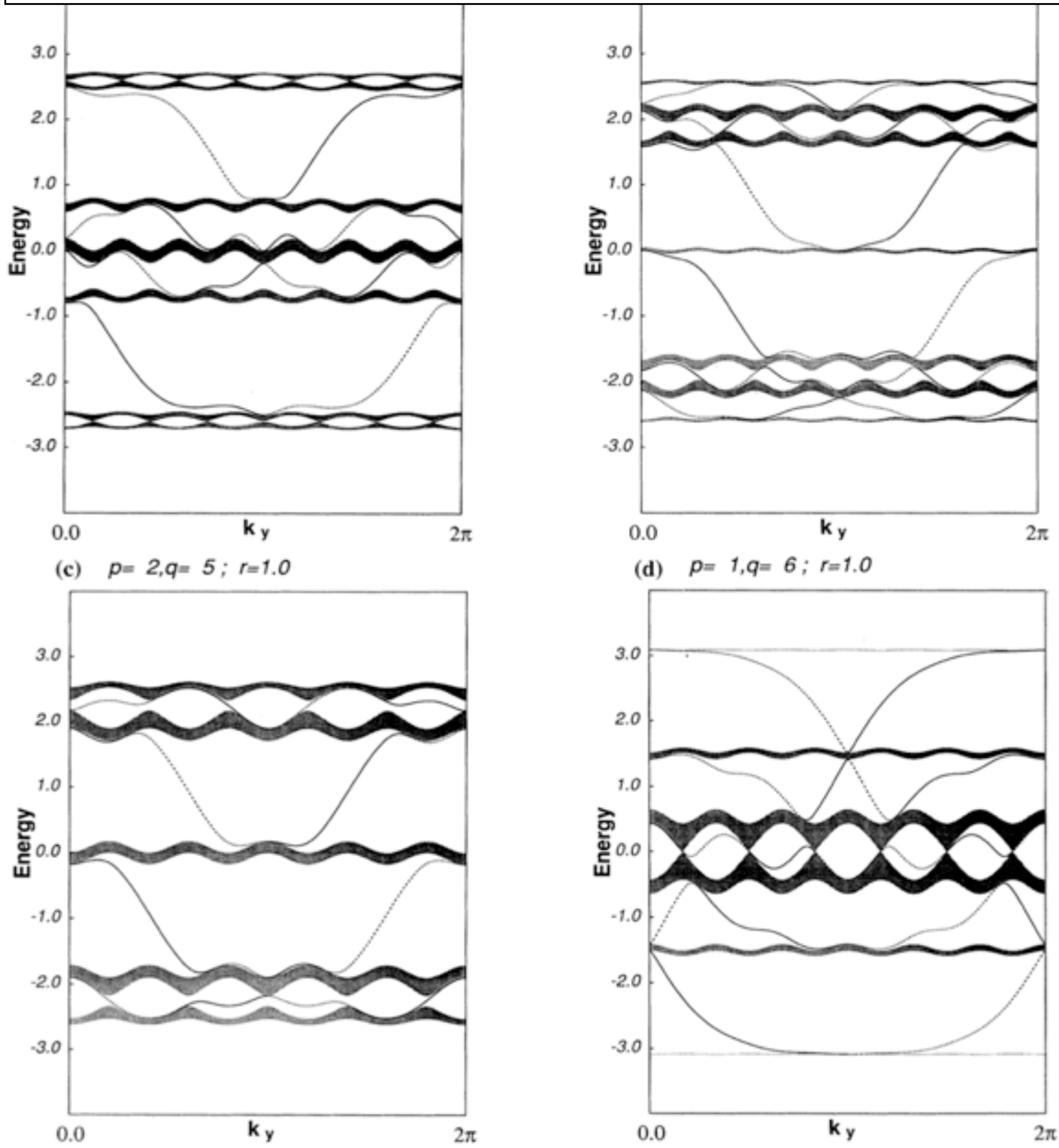
Experimentally realized

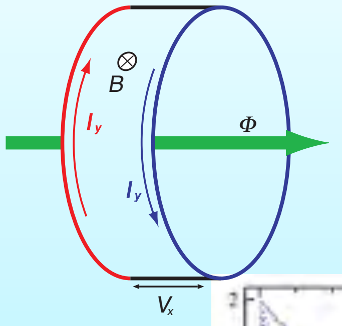
Quantum Hall edge states : Lattice electrons on cylinder

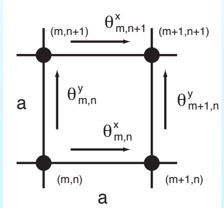
Topol. char. by edges

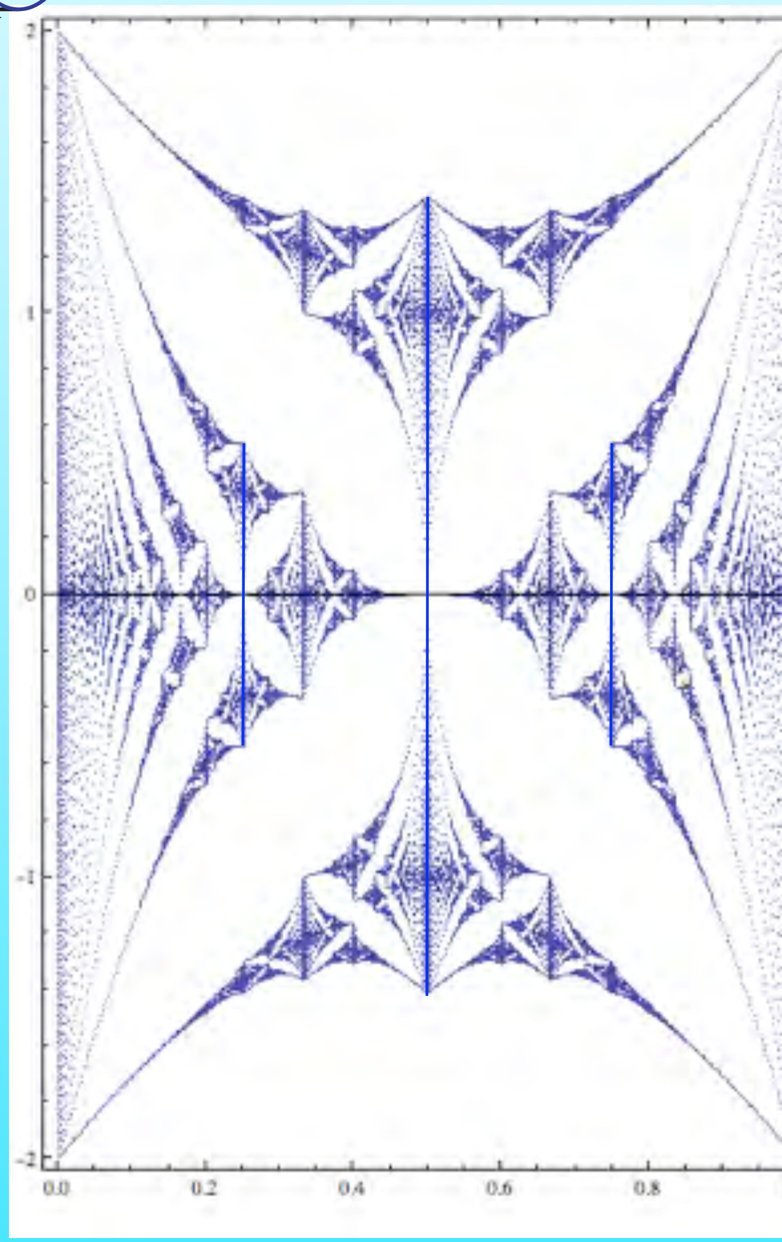
In gap states as the edge states

Y. H, Phys. Rev. B 48, 11851–11862 (1993)









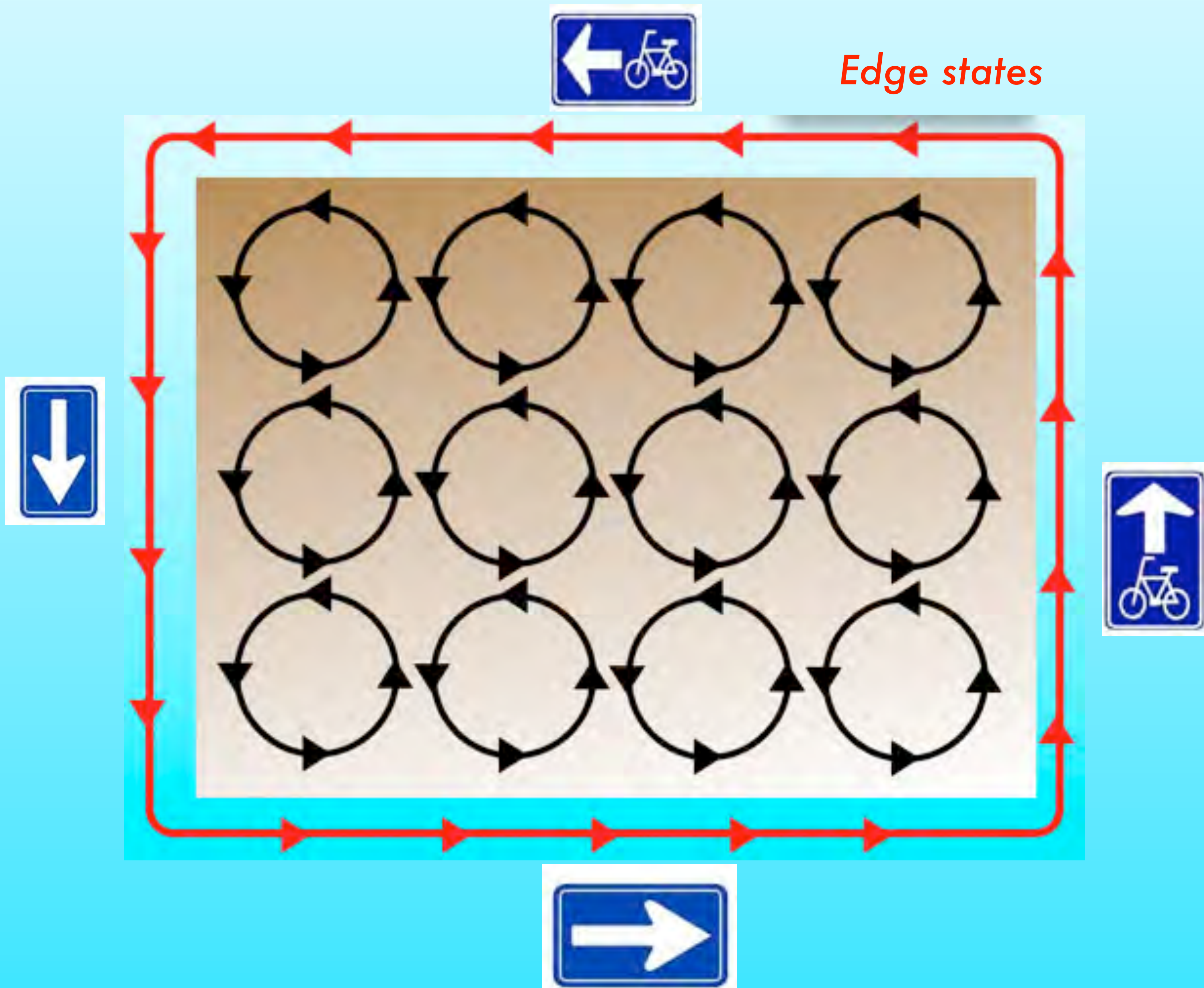
C. Albrecht, J.H. Smet, K. von Klitzing, D. Weiss, V. Umansky, H. Schweizer: Evidence of Hofstadter's Fractal Energy Spectrum in the Quantized Hall Conductance. Phys. Rev. Lett. 86(1), 147-150 (2001).

Experimentally realized

Edge states are topologically stable

Cyclotron motion by Lorentz force $\boldsymbol{F} = -e\boldsymbol{v} \times \boldsymbol{B}$

Currents are canceled in the bulk but induces a boundary current



Edge states are chiral

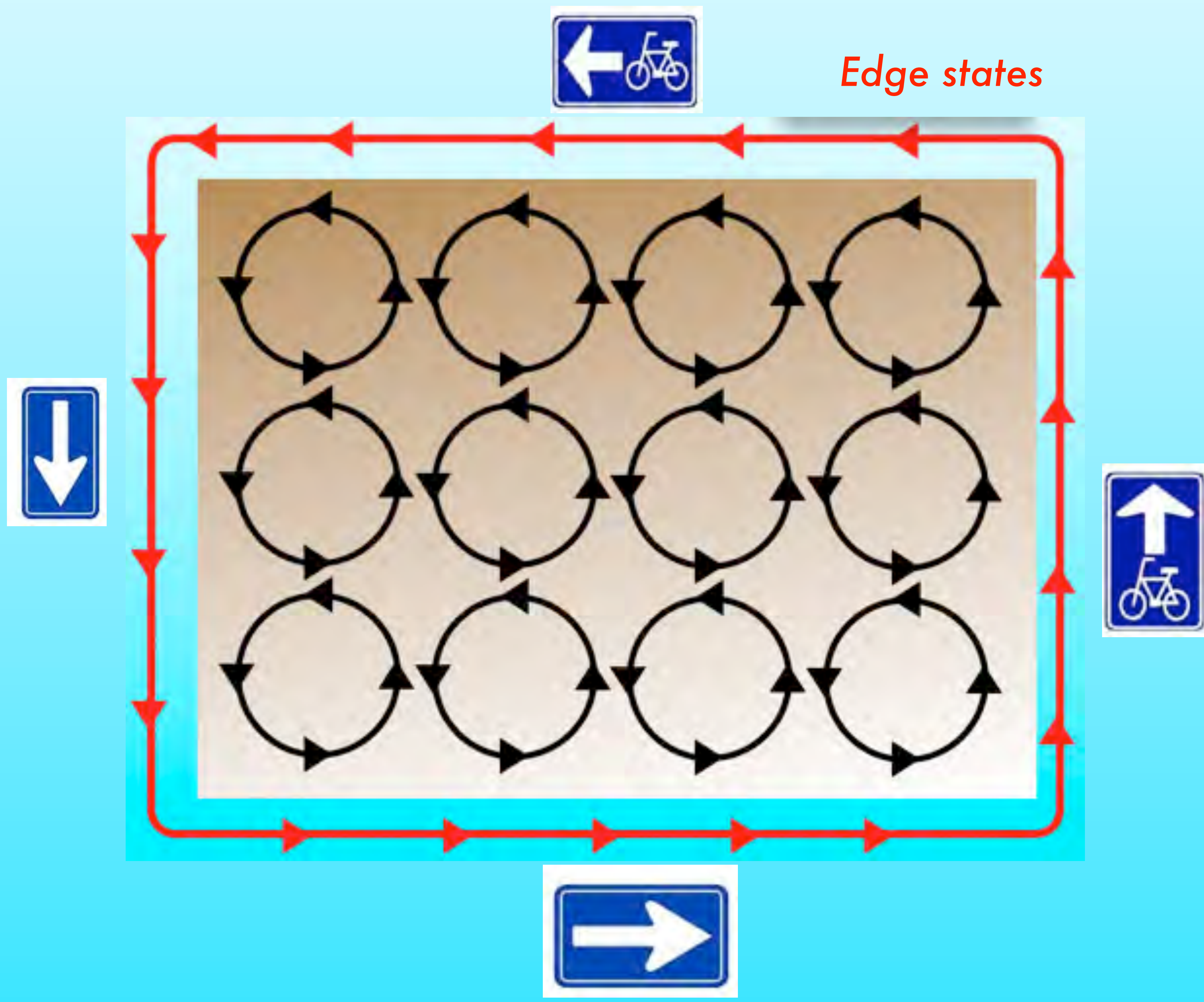
One way going !!

Cannot stop !

Edge states are topologically stable

Cyclotron motion by Lorentz force $\boldsymbol{F} = -e\boldsymbol{v} \times \boldsymbol{B}$

Currents are canceled in the bulk but induces a boundary current



Edge states are chiral

One way going !!

Cannot stop !

No back scattering

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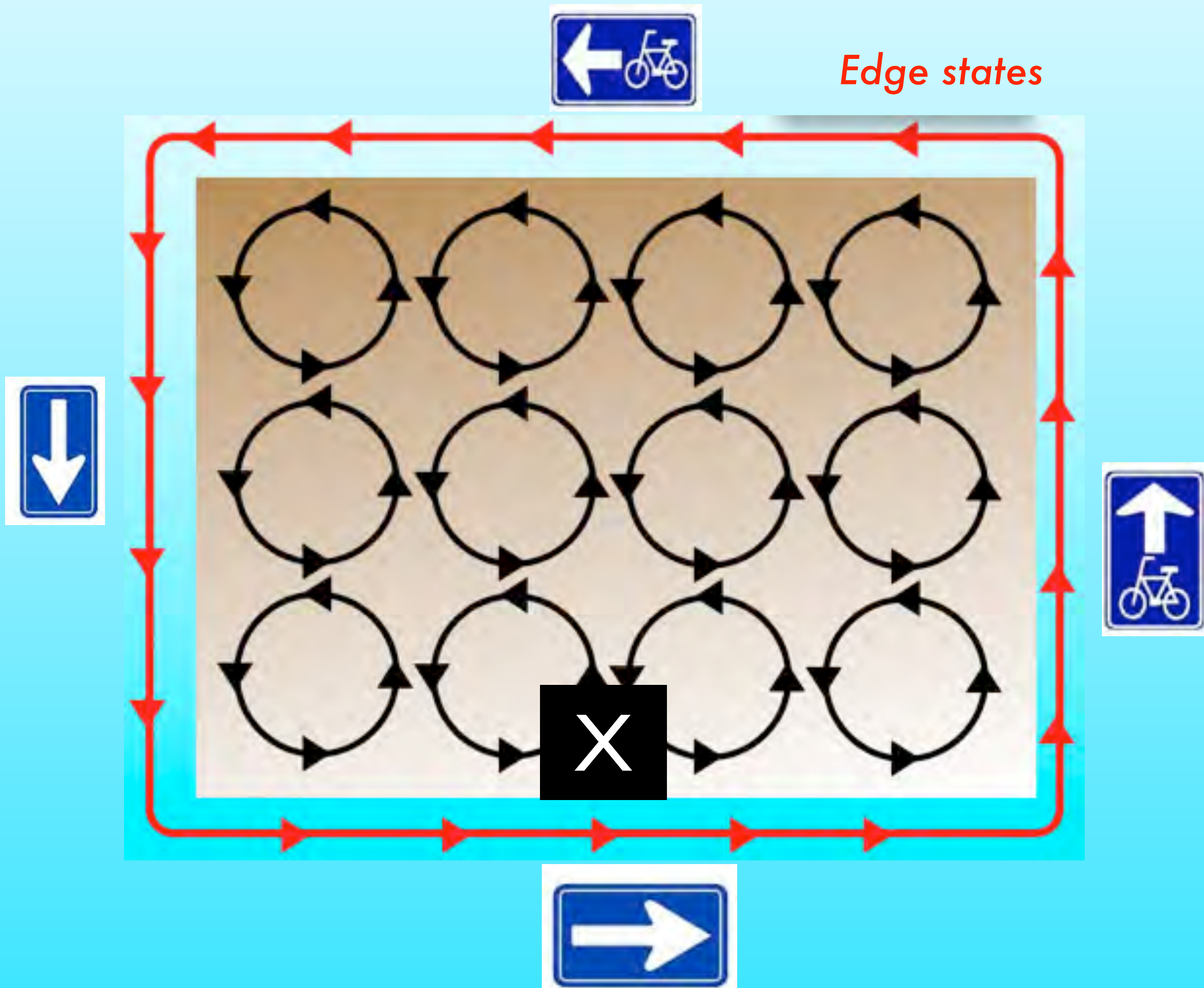

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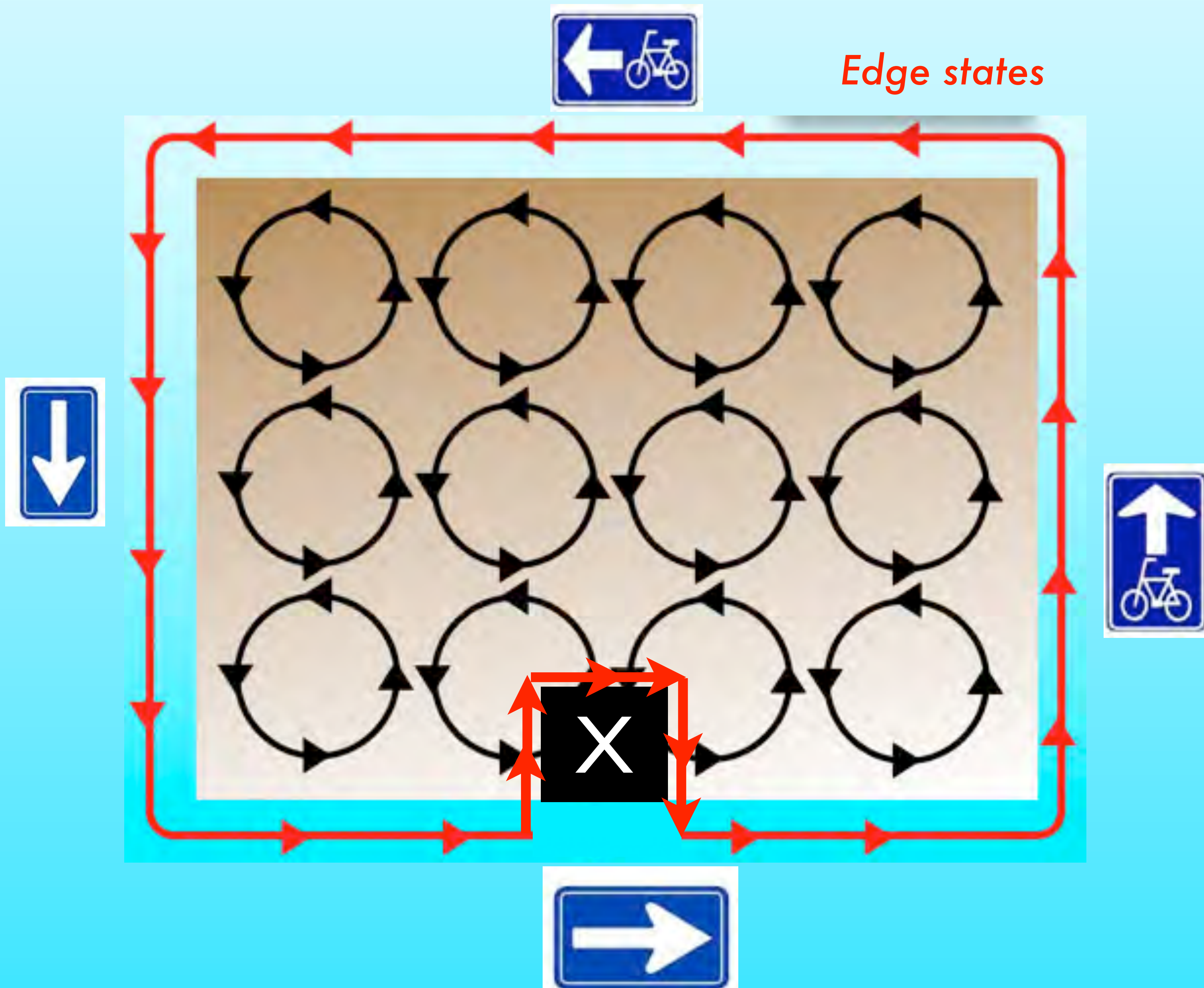

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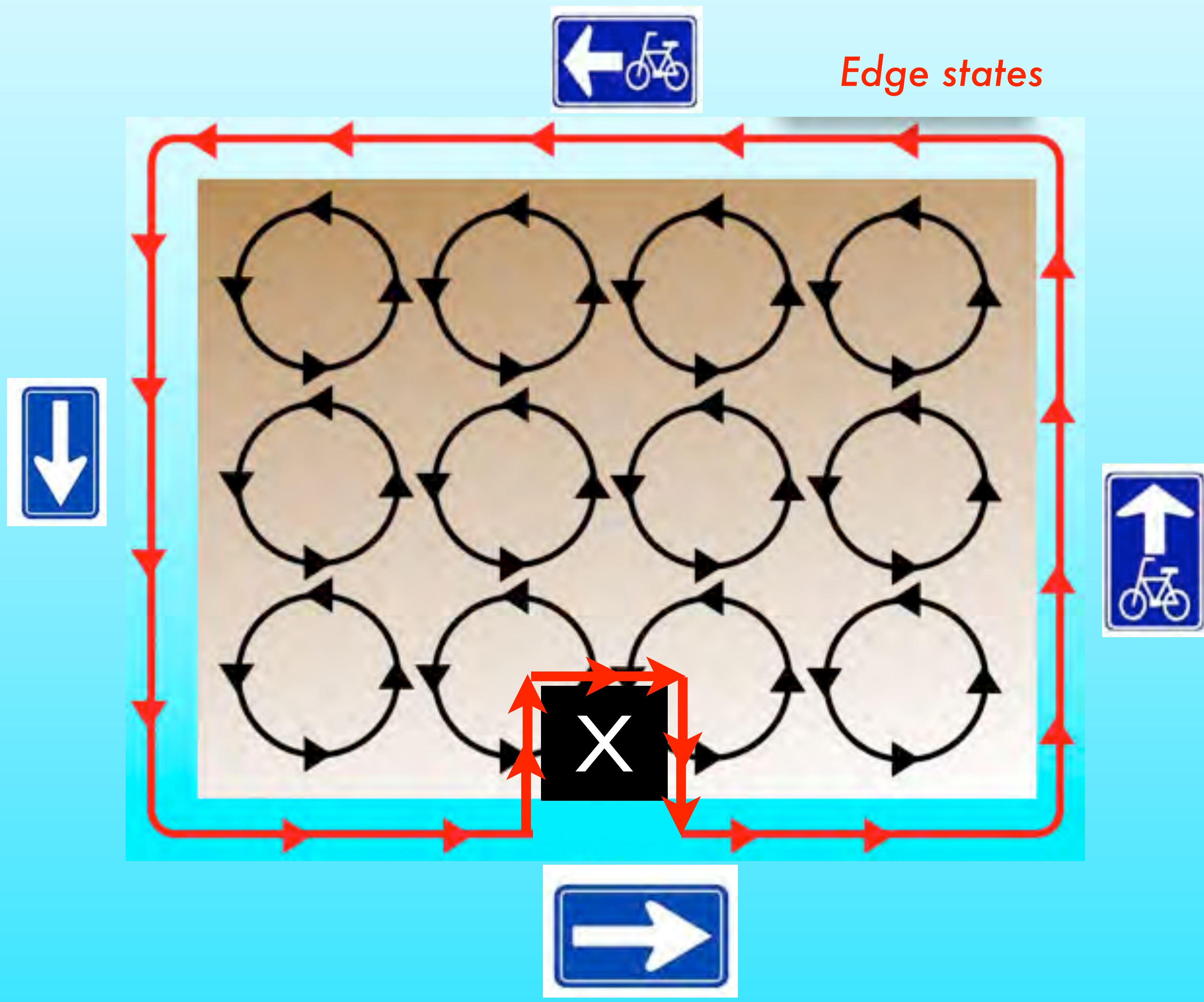


Edge states are *chiral*

One way going !!

Cannot stop !

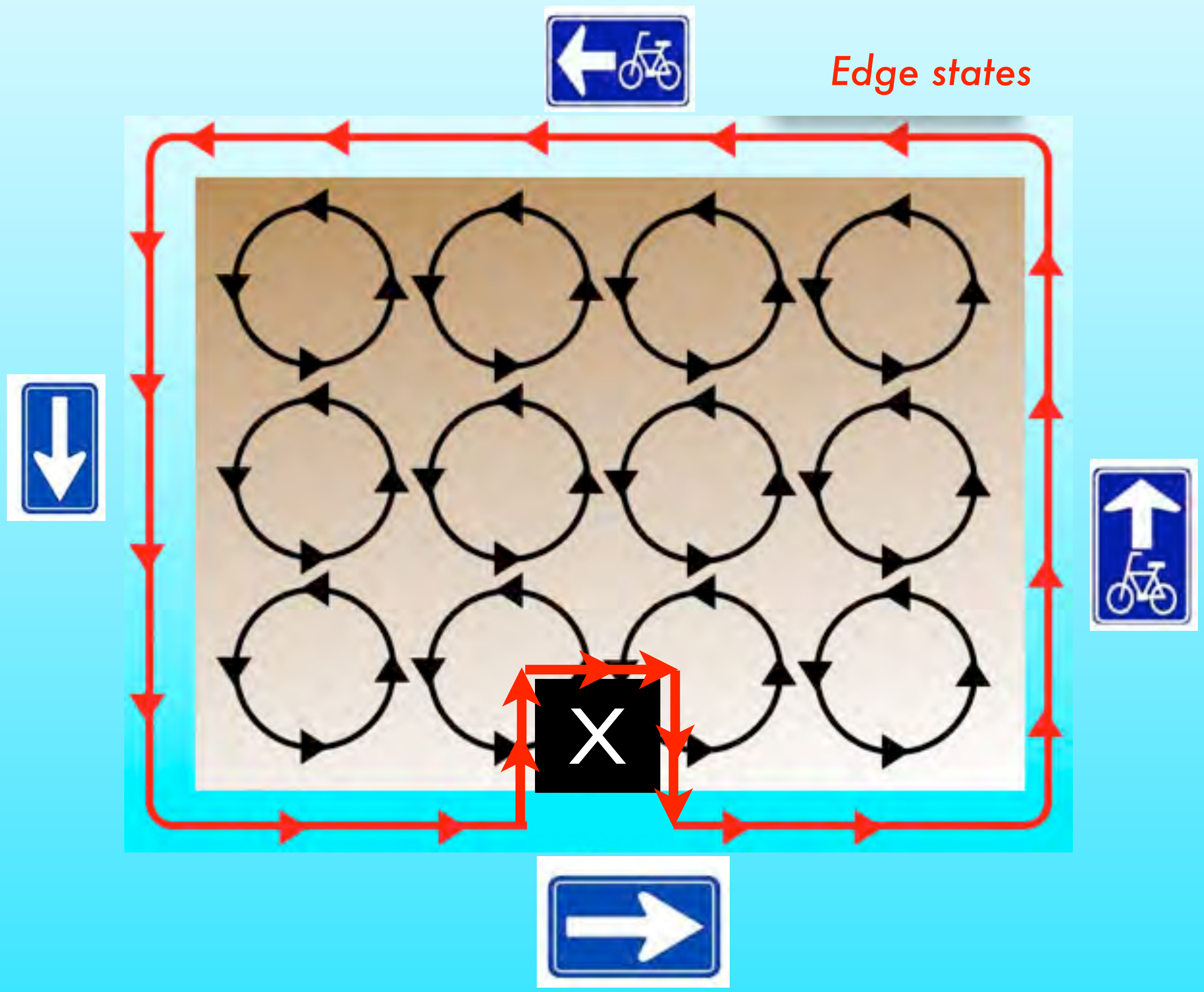
No back scattering

Stable for impurities !!

Edge states are topologically stable

Cyclotron motion by Lorentz force $\boldsymbol{F} = -e\boldsymbol{v} \times \boldsymbol{B}$

Currents are canceled in the bulk but induces a boundary current



Edge states are chiral

One way going !!

Cannot stop !

No back scattering

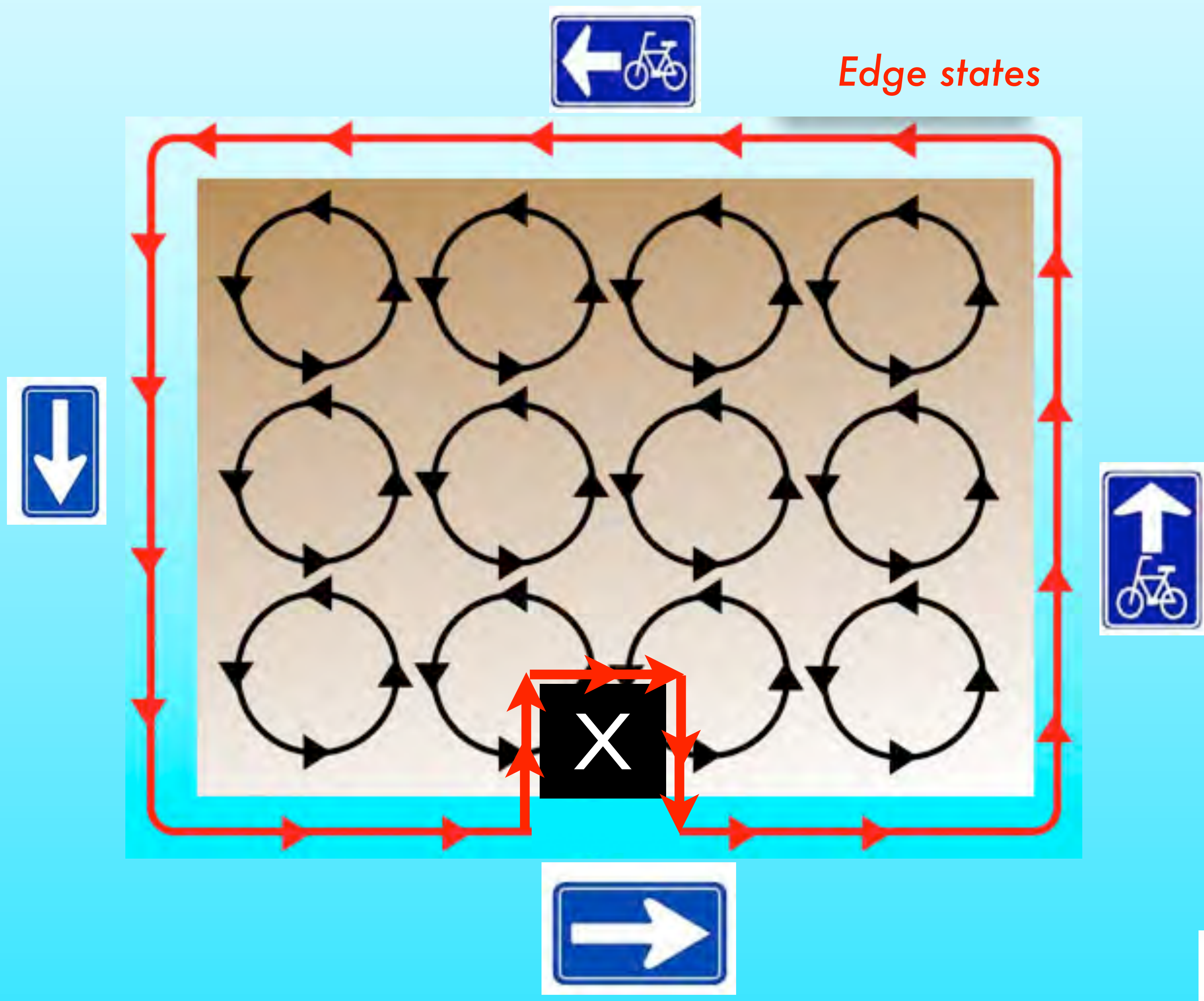
Stable for impurities !!

Topological stability
of
Chiral edge states

Edge states are topologically stable

Cyclotron motion by Lorentz force $\boldsymbol{F} = -e\boldsymbol{v} \times \boldsymbol{B}$

Currents are canceled in the bulk but induces a boundary current



Edge states are chiral

One way going !!

Cannot stop !

No back scattering

Stable for impurities !!

Topological stability
of
Chiral edge states

Edge States of Graphene

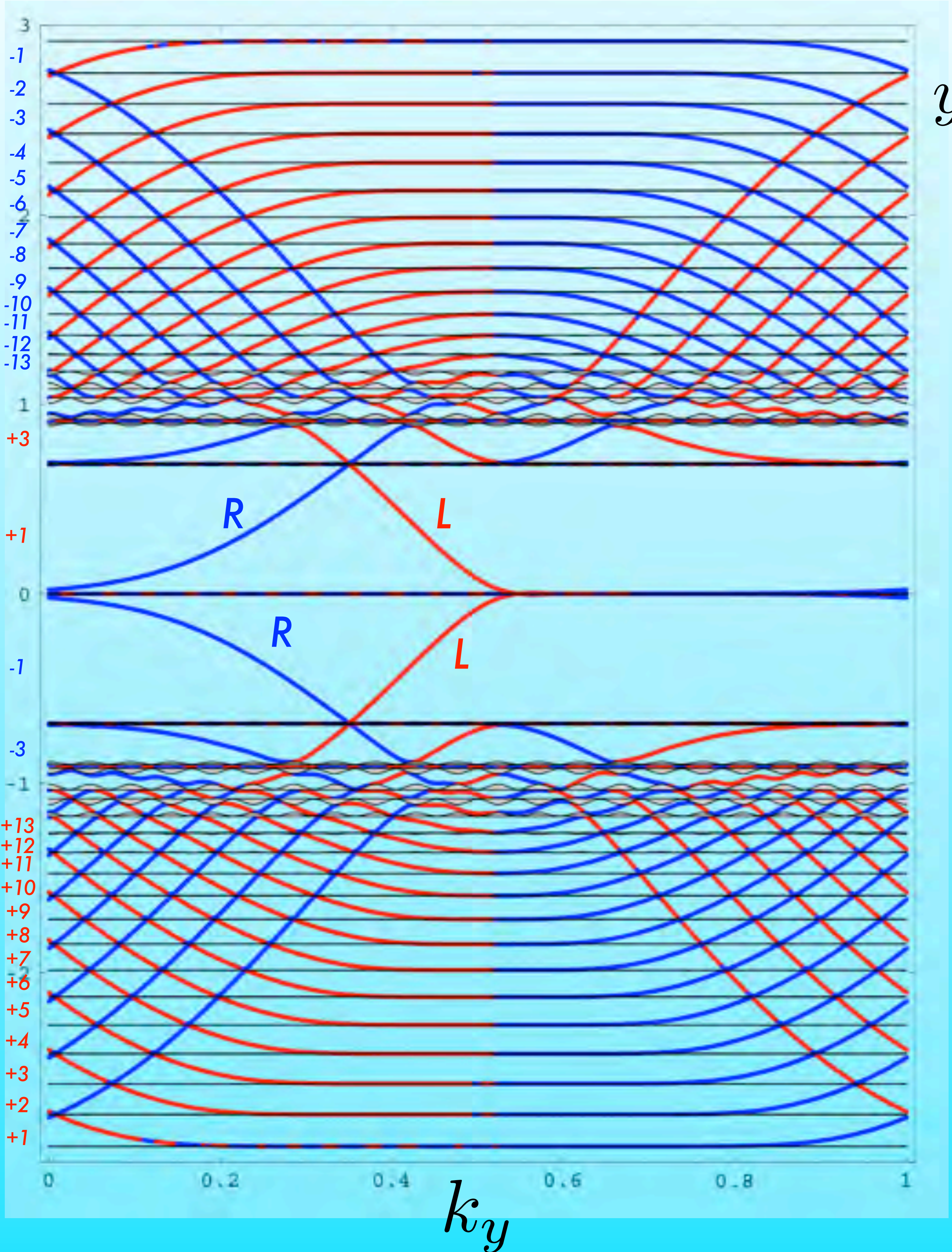
Topol. char. by edges

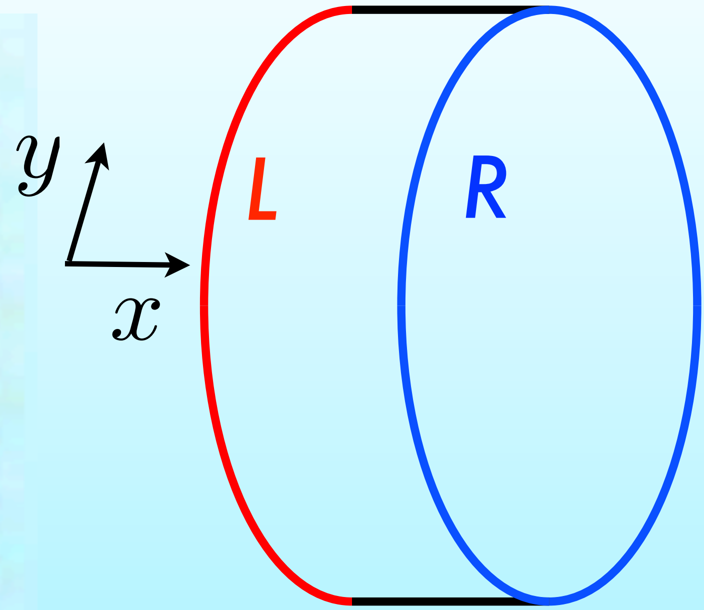
Standard Quantization (hole)

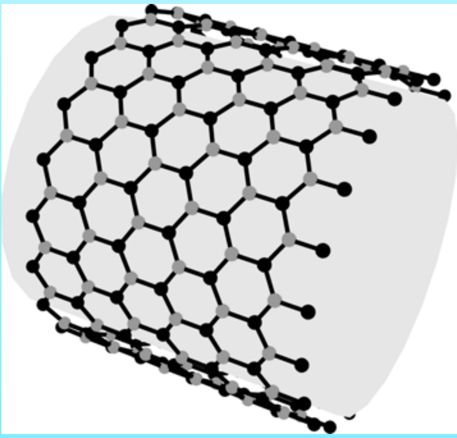
$\phi = 1/21$

Dirac Type Quantization

Standard Quantization





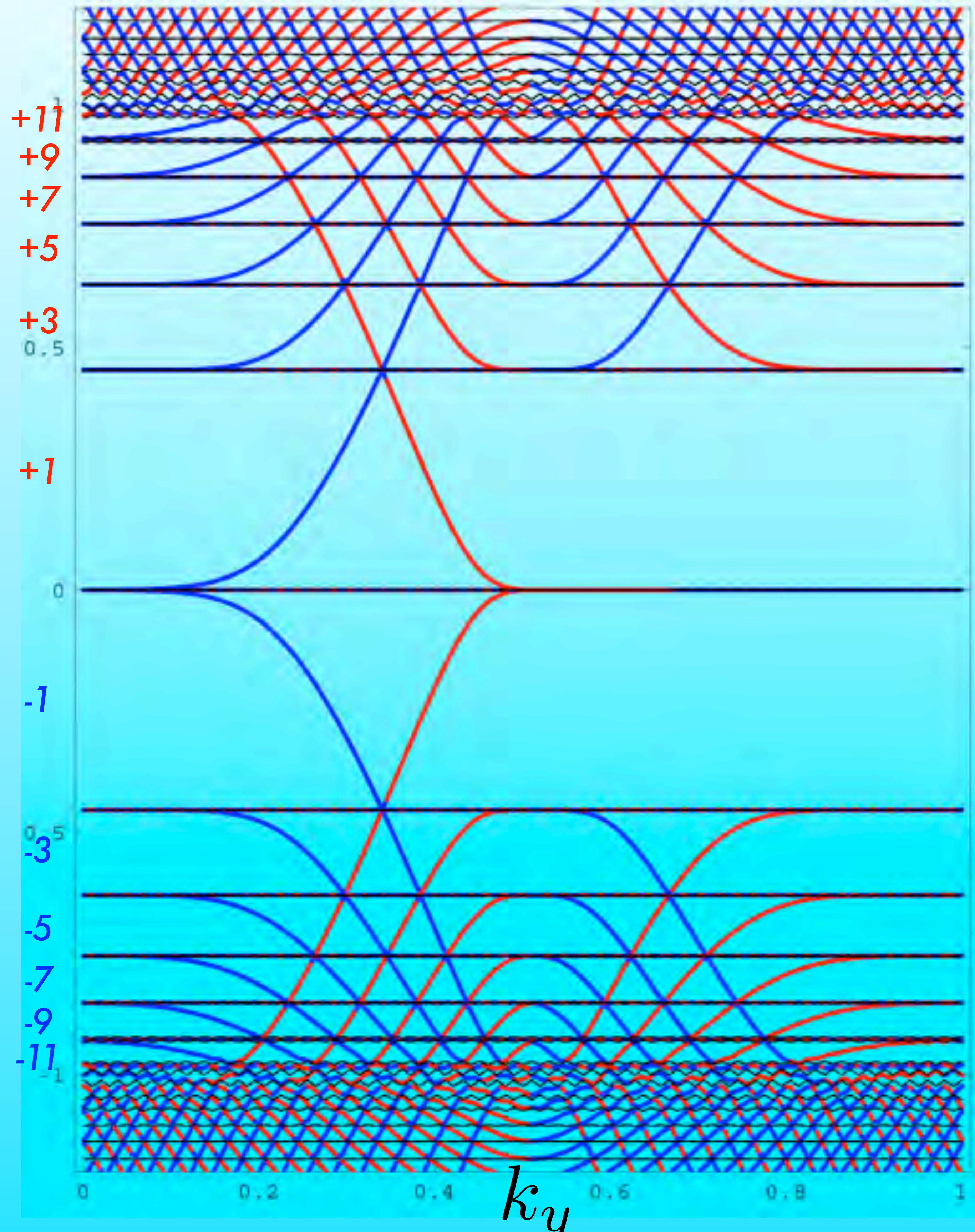


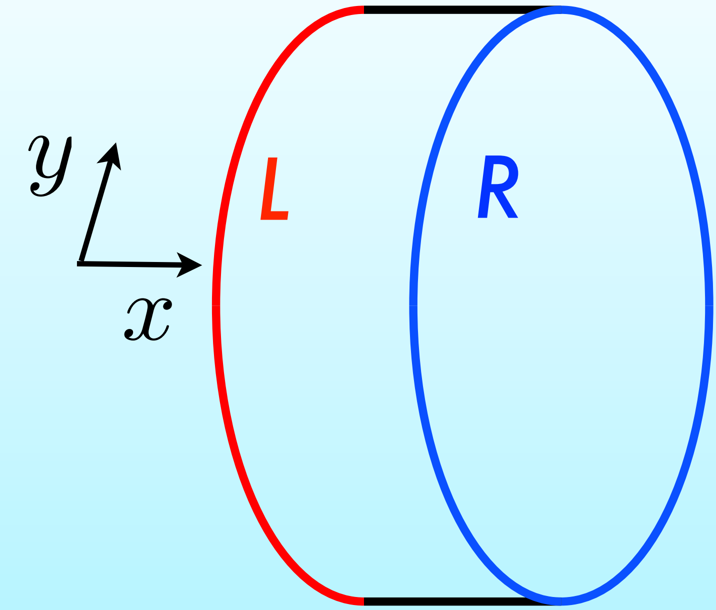
Edge States of Graphene

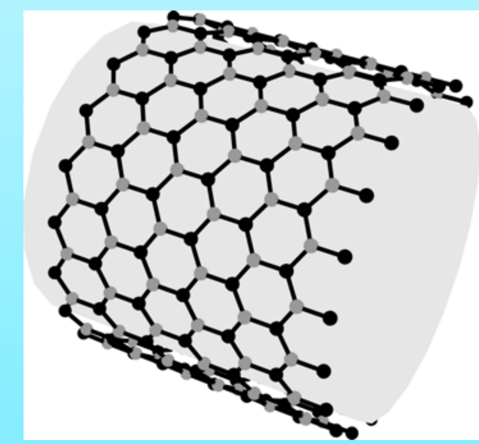
Topol. char. by edges

$\phi = 1/51$

Edge States being consistent with Dirac Type Quantization

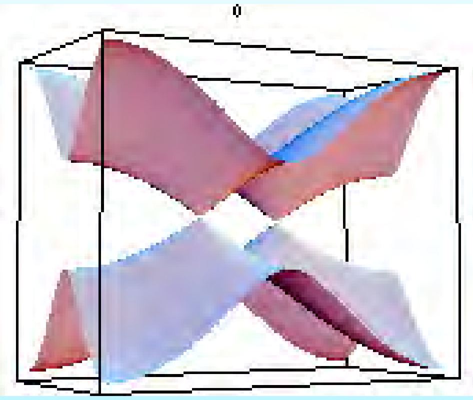


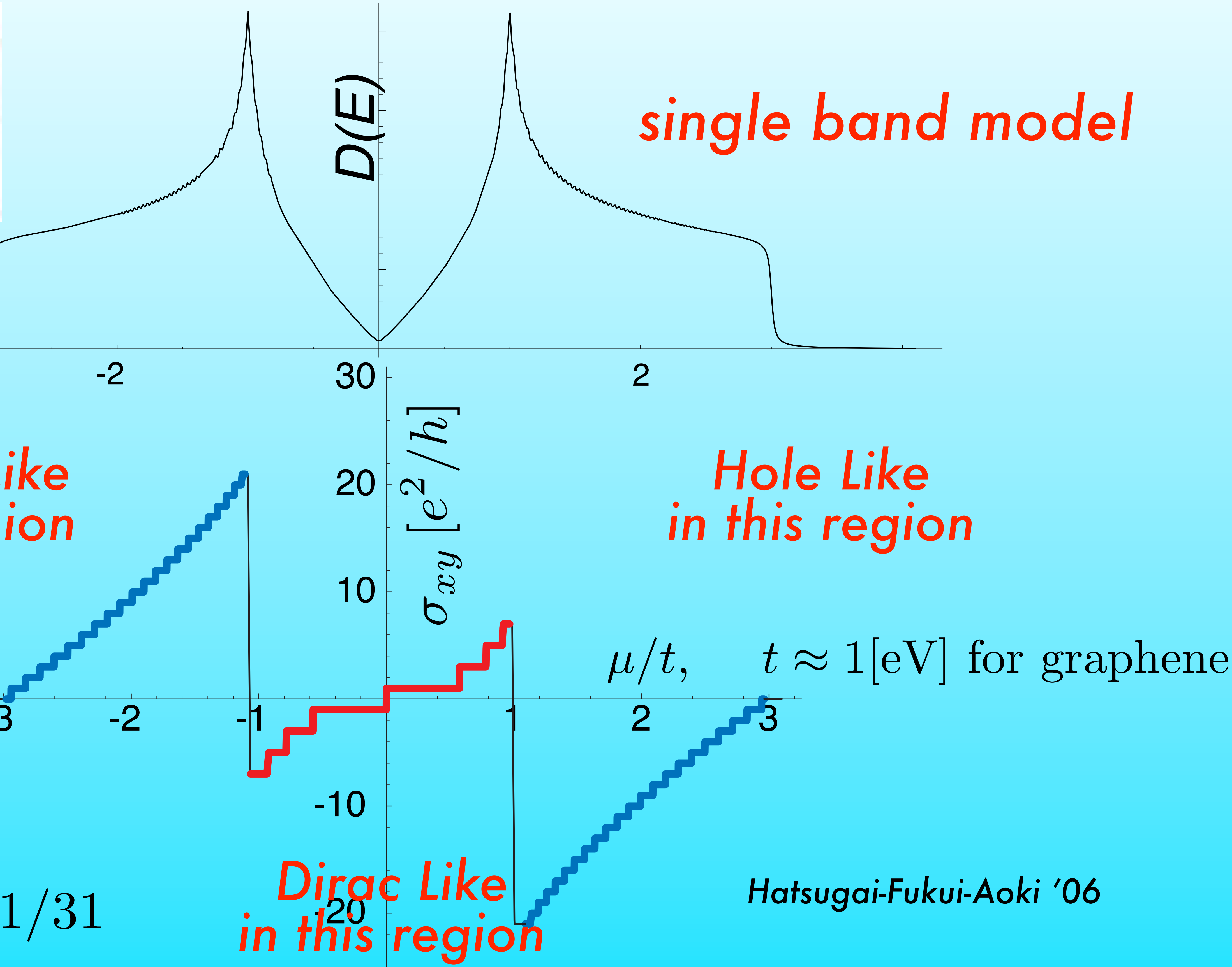




Hall Conductace vs chemical potential

- ☆ Accurate Hall conductance over whole spectrum





Hall Conductace vs chemical potential

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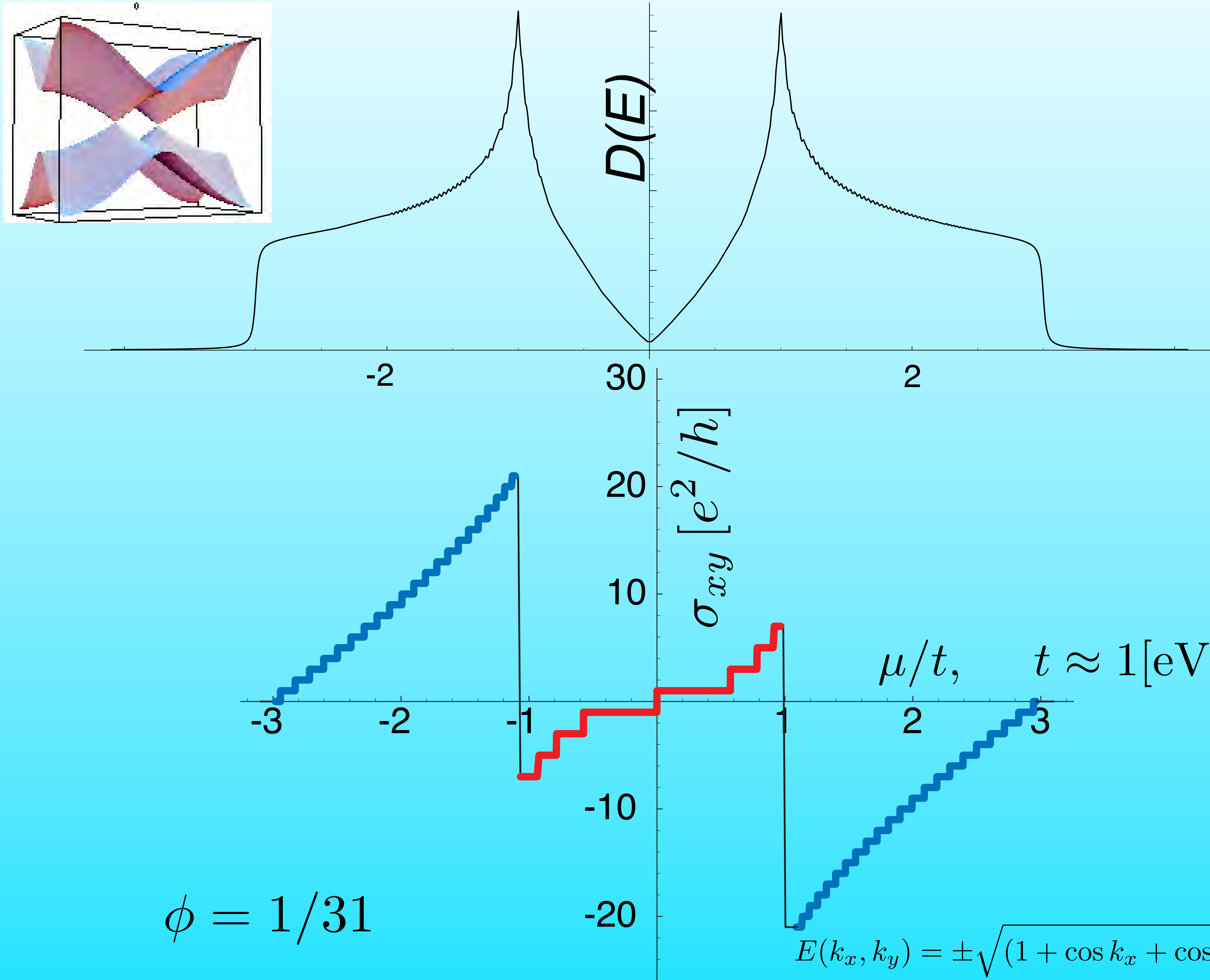


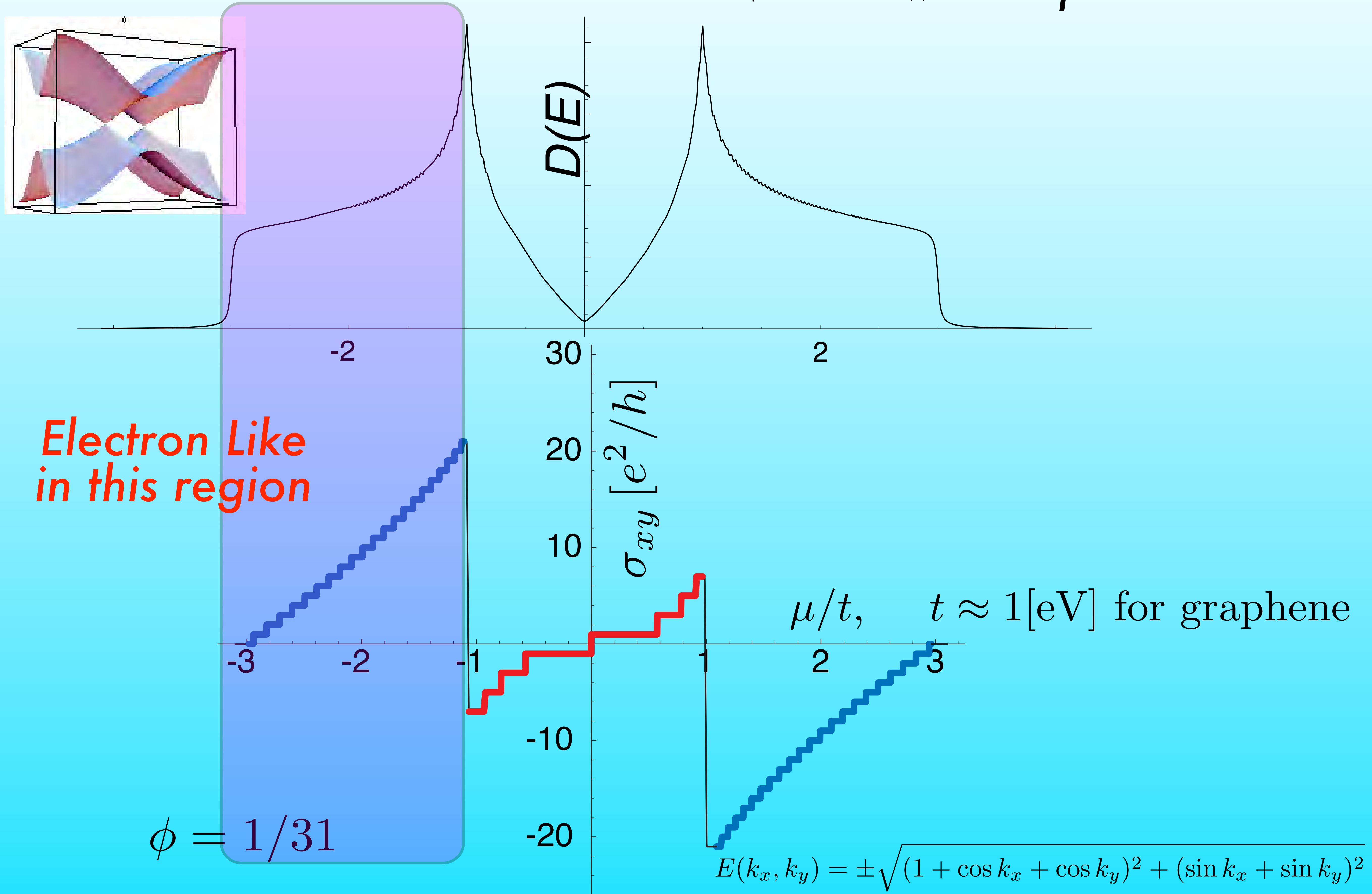
$\mu/t, \quad t \approx 1[\mathrm{eV}]$ for graphene

$\phi = 1/31$

$$E(k_x, k_y) = \pm\sqrt{(1+\cos k_x + \cos k_y)^2 + (\sin k_x + \sin k_y)^2}$$

Hall Conductace vs chemical potential

- Accurate Hall conductance over the whole spectrum



Hall Conductace vs chemical potential

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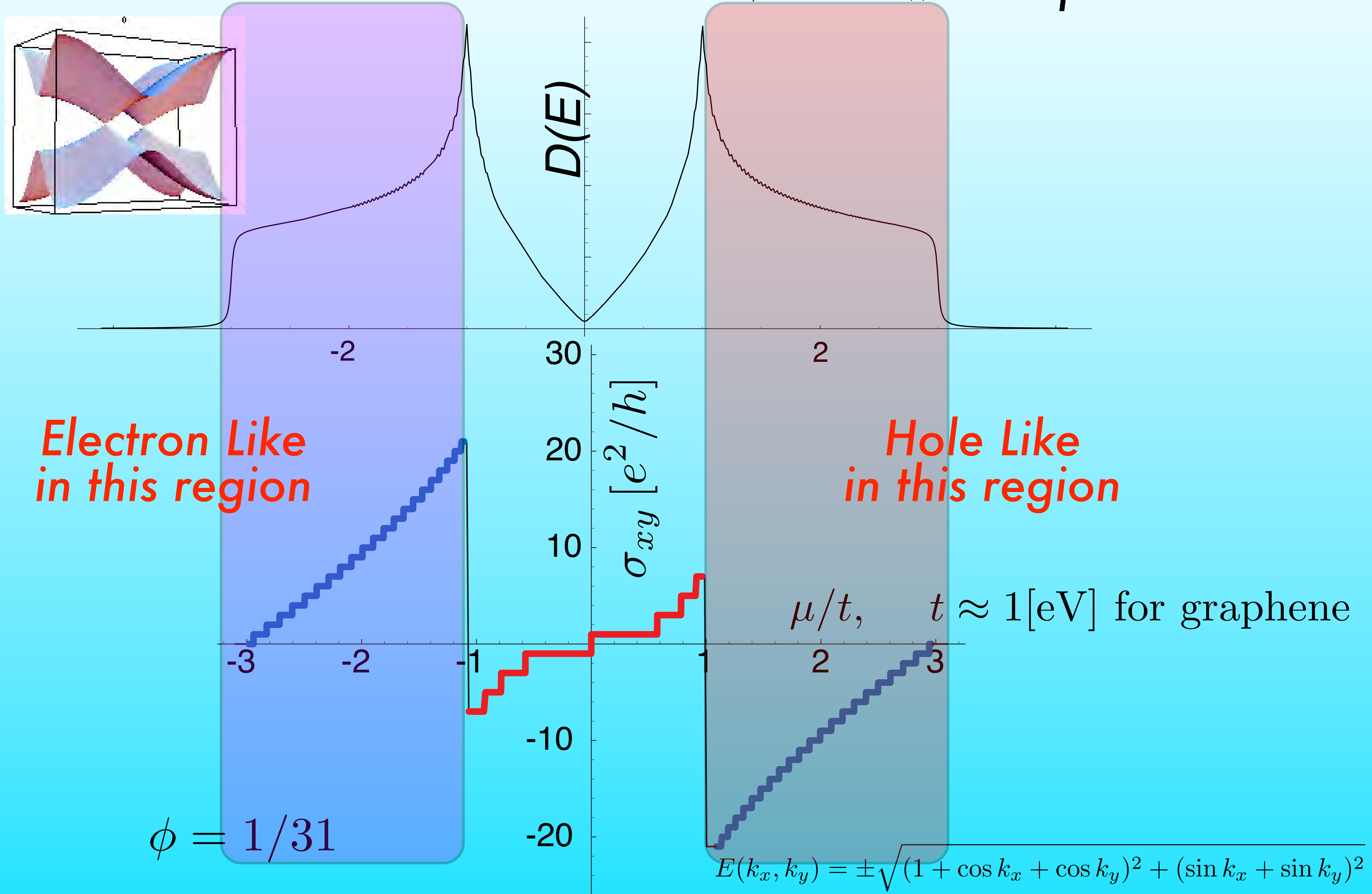
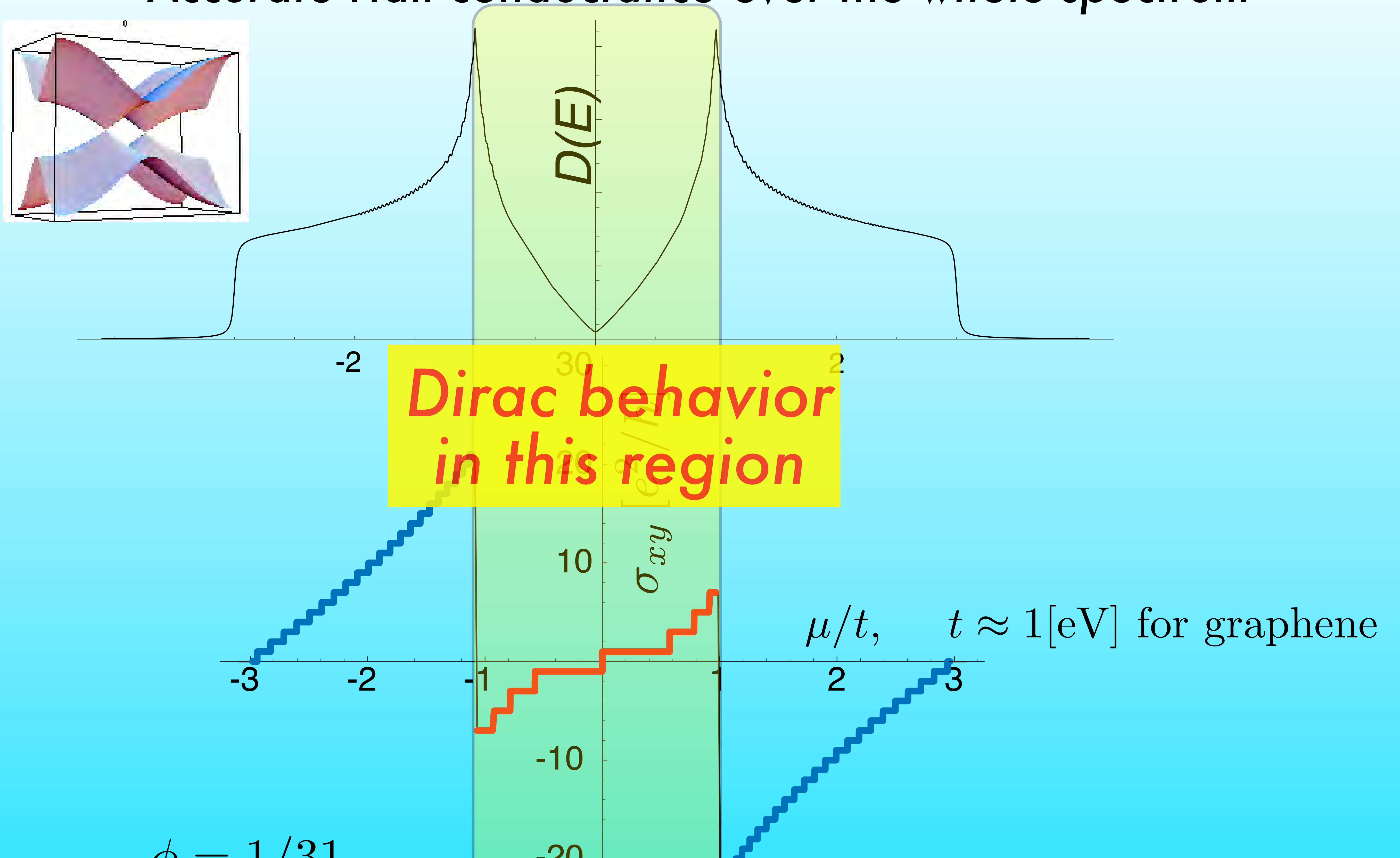


Hall Conductace vs chemical potential

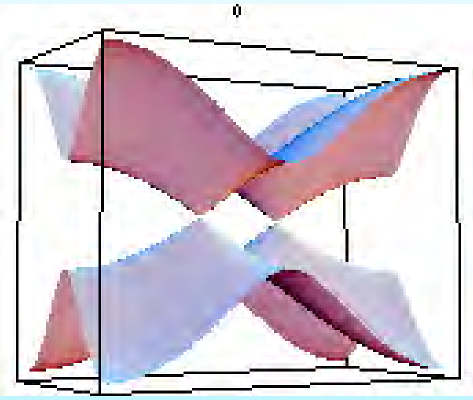
- Accurate Hall conductance over the whole spectrum

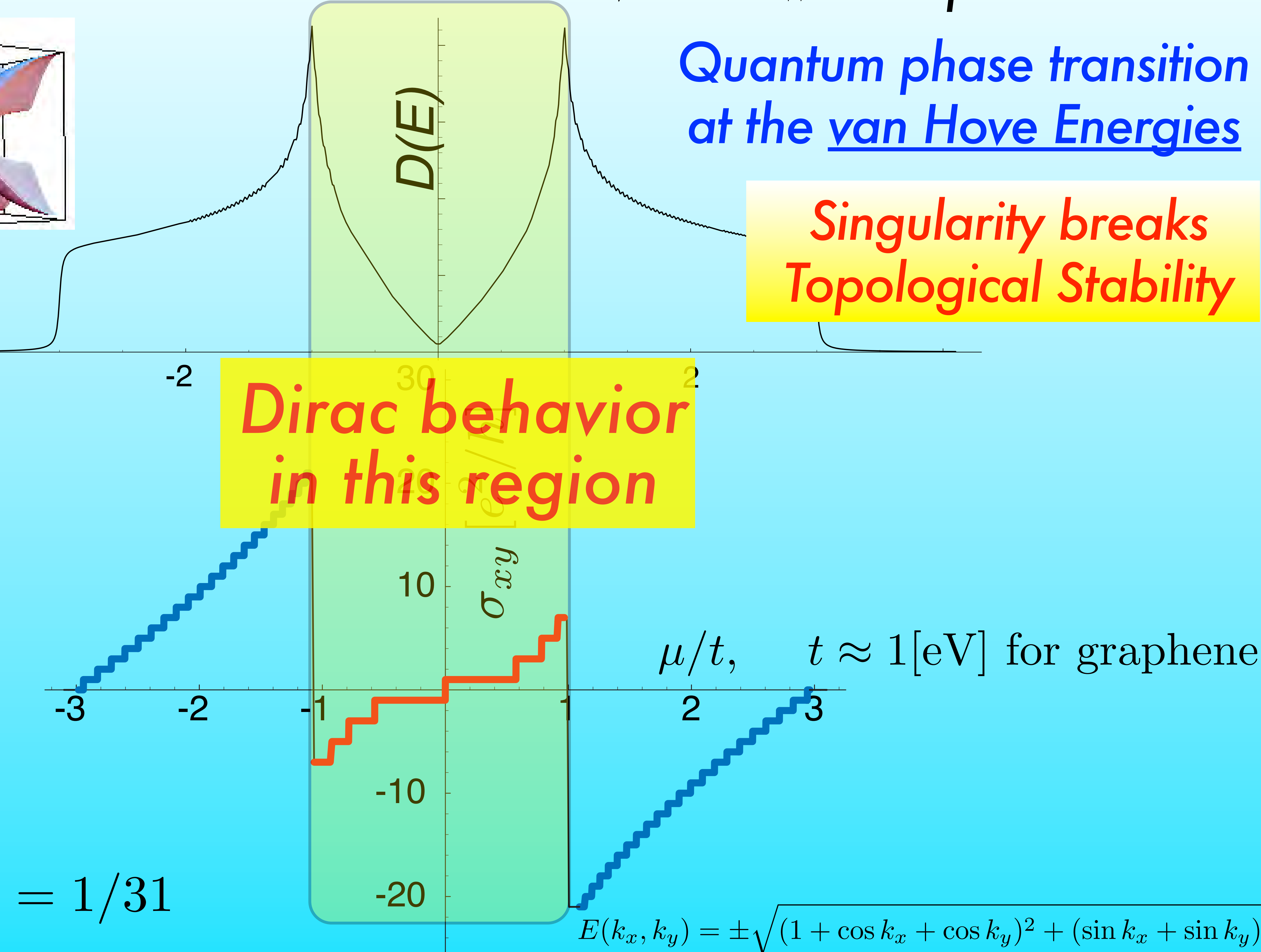


$$E(k_x, k_y) = \pm\sqrt{(1 + \cos k_x + \cos k_y)^2 + (\sin k_x + \sin k_y)^2}$$

Hall Conductace vs chemical potential

- ☆ Accurate Hall conductance over the whole spectrum





How the edge states determine ?

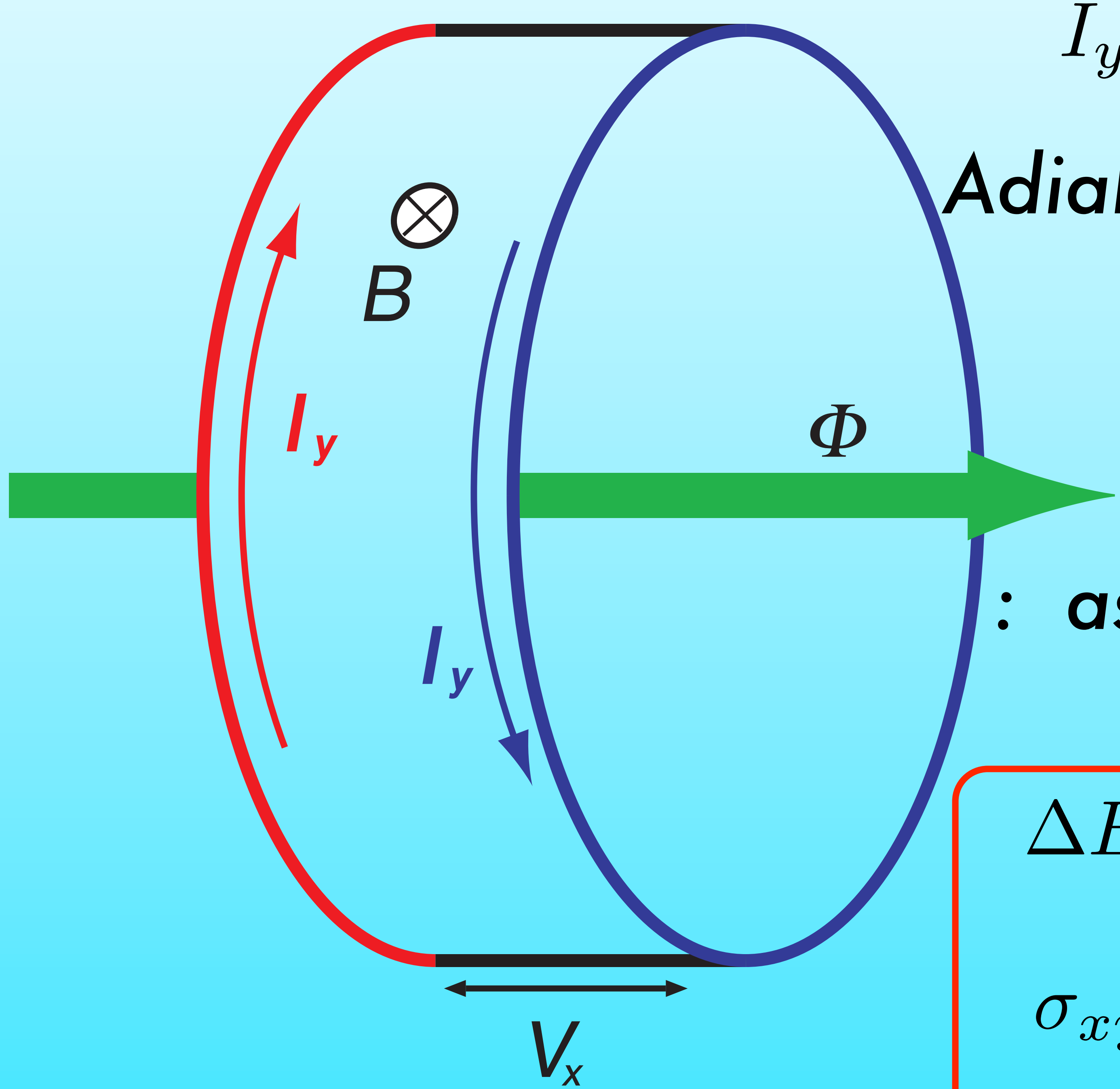
How to calculate σ_{xy} by the edge states?

Laughlin's Argument & Edge States

Topol. char. by edges

☆ Gauge Invariance & Byers-Yang' Formula

Laughlin '81



$$I_y = \frac{\Delta E}{\Delta \Phi} = \sigma_{xy} V_x$$ Byers-Yang

Adiabatic increase by $\Delta\Phi = \Phi_0 = \frac{h}{e}$

→ Insulating System goes back to the Original State

: assume n electrons are carried from the left to the right

$$\boxed{\begin{aligned} \Delta E &= n e V_x \\ \sigma_{xy} &= \frac{e^2}{h} n \end{aligned}}$$

n is an integer but unknown

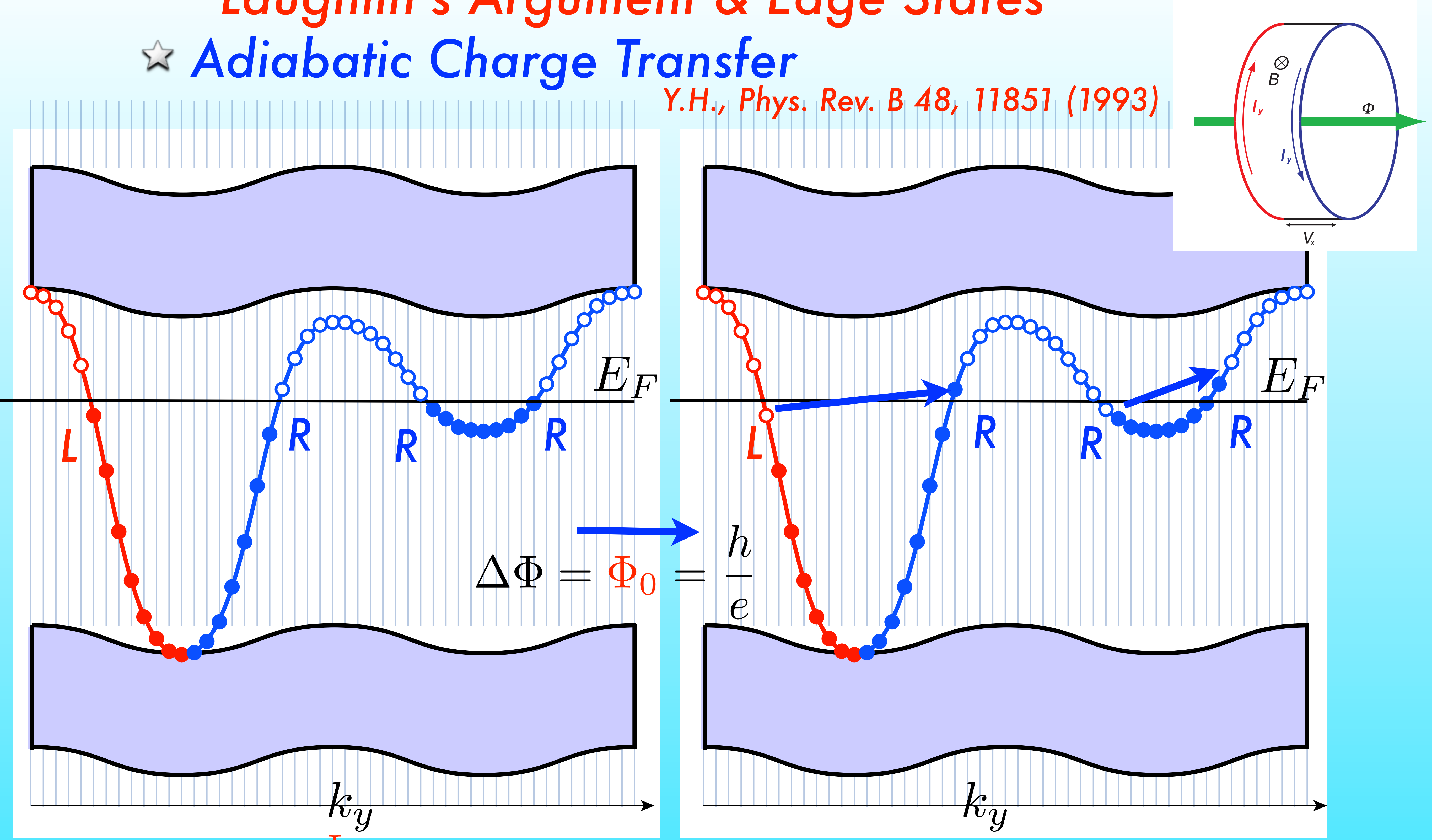
Quantization of σ_{xy} by Edge states

Topol. char. by edges

Laughlin's Argument & Edge States

☆ Adiabatic Charge Transfer

Y.H., Phys. Rev. B 48, 11851 (1993)

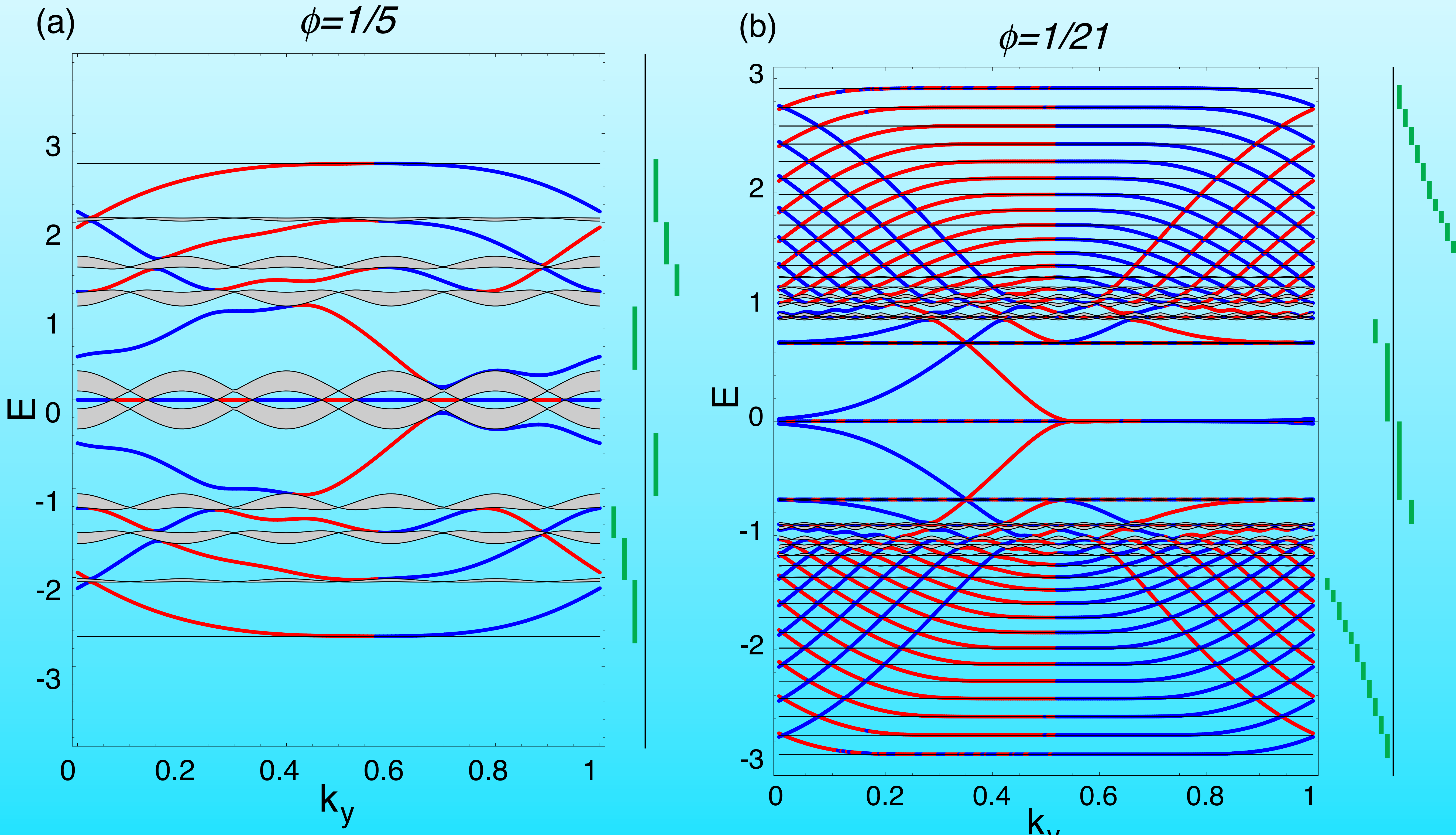


$$k_y = 2\pi \frac{n + \frac{\Phi}{\Phi_0}}{L_y}, \quad n: \ \text{integers}$$

1 Electron is carried from the Left to the right in this case

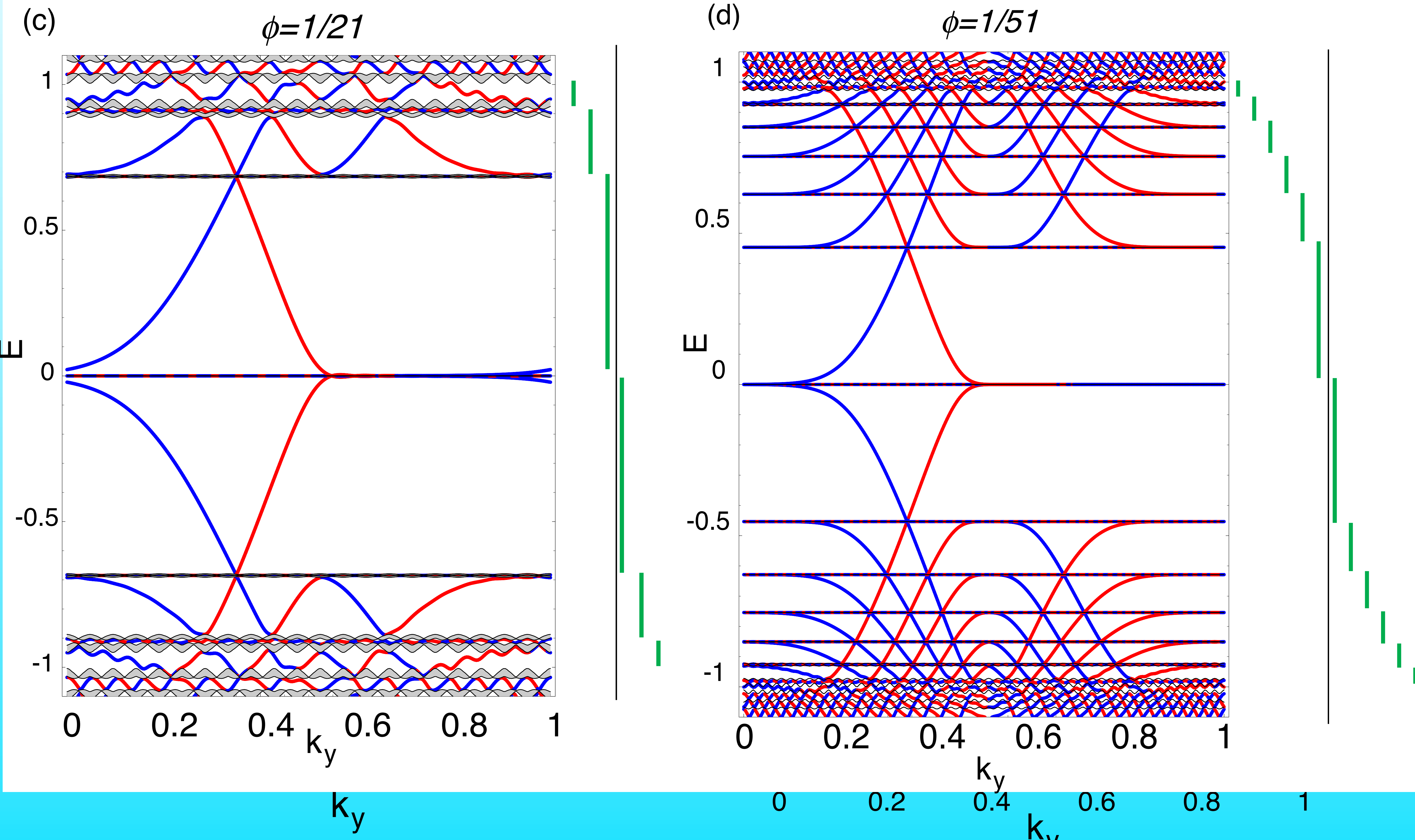
$$\sigma_{xy} = \frac{e^2}{h} \cdot 1$$

Bulk – Edge Correspondence ?



Bulk – Edge Correspondence ?

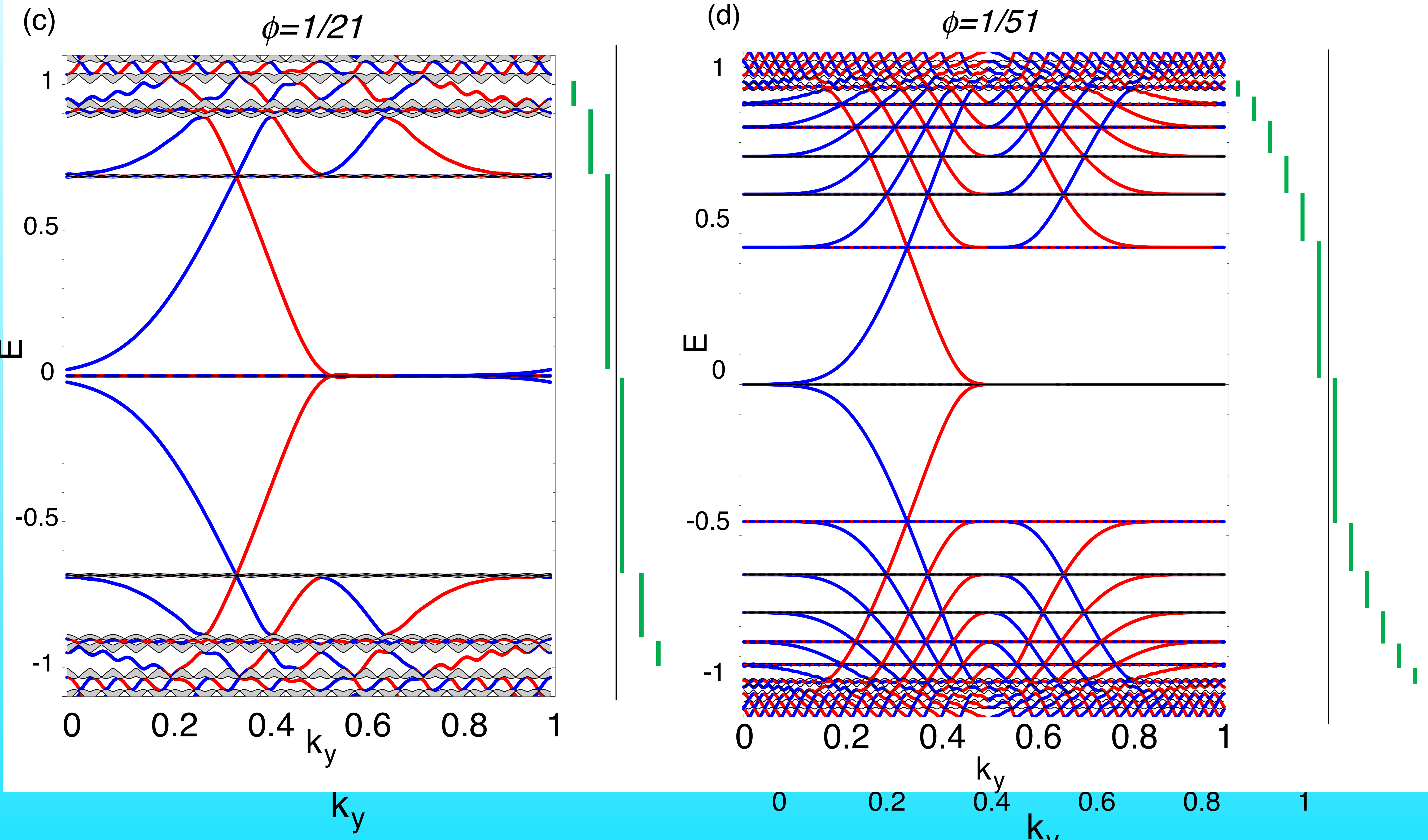
Near Zero



Bulk – Edge Correspondence ?

☆ *Numerically* $\sigma_{xy}{}^{\text{bulk}} = \sigma_{xy}{}^{\text{edge}}$

Near Zero

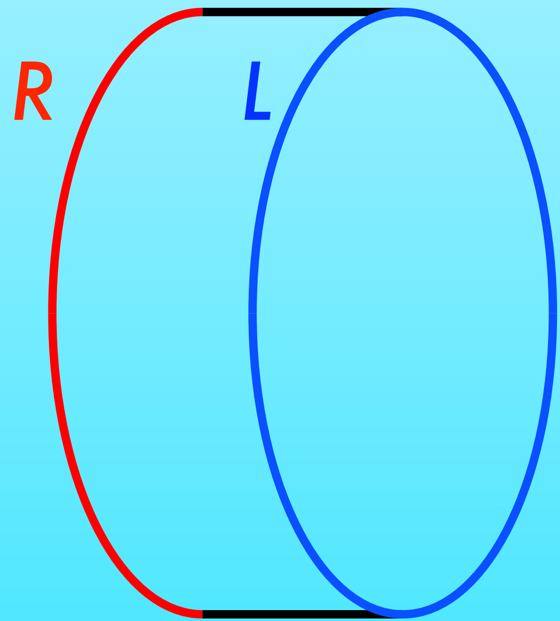


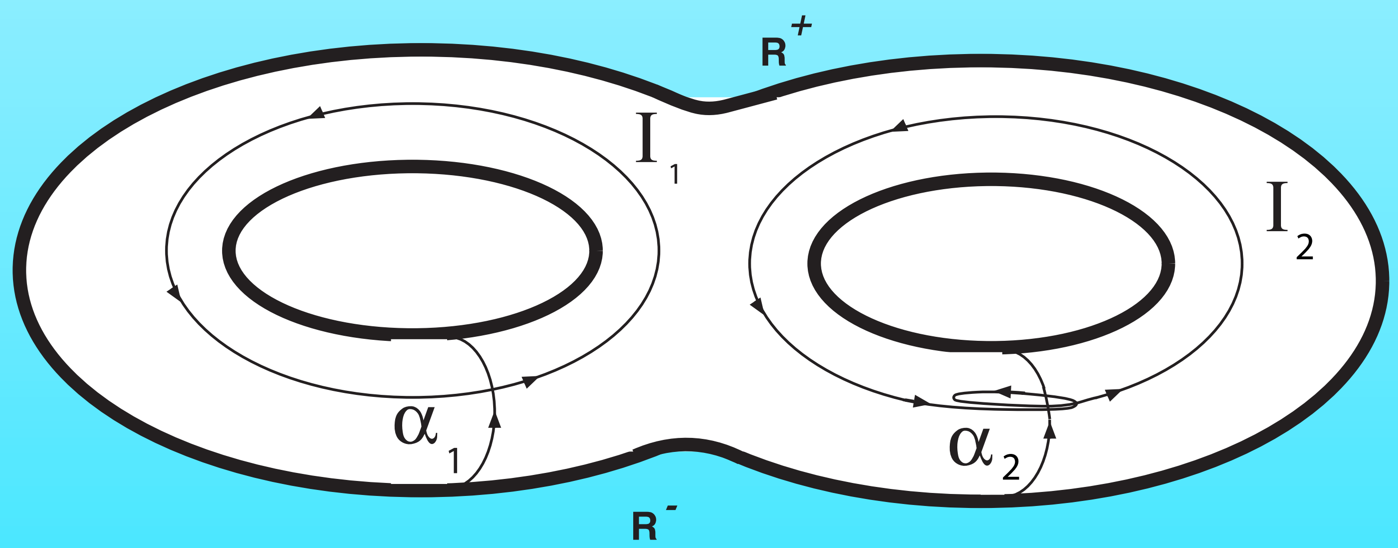
Analytical Consideration of Edge states in Graphene

YH, T. Fukui & H. Aoki, Phys. Rev. B74, 205414 (2006)

☆ Followed by the discussion on a square lattice

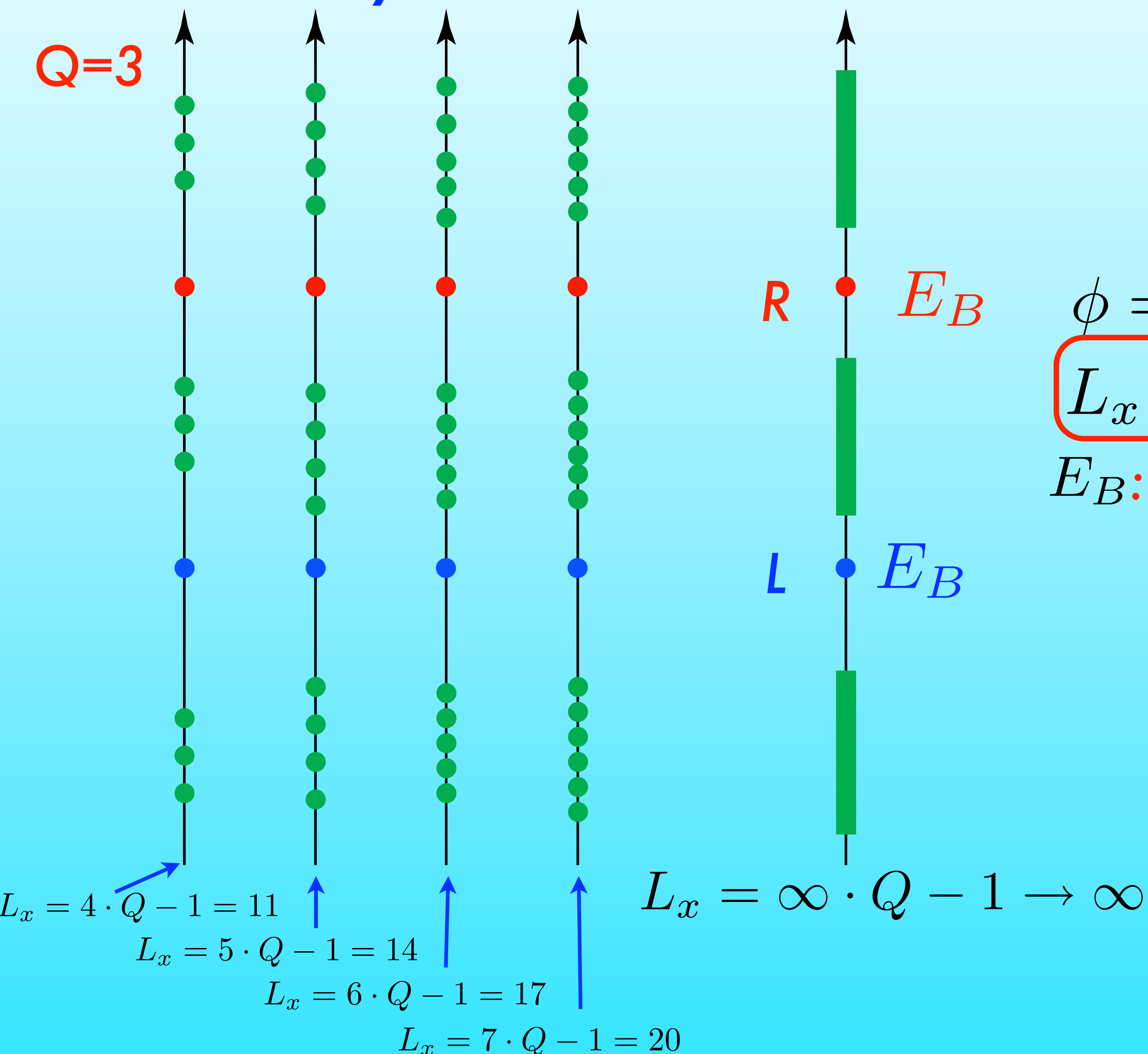
Y.H., Phys. Rev. B 48, 11851 (1993)
Phys. Rev. Lett. 71, 3697 (1993)

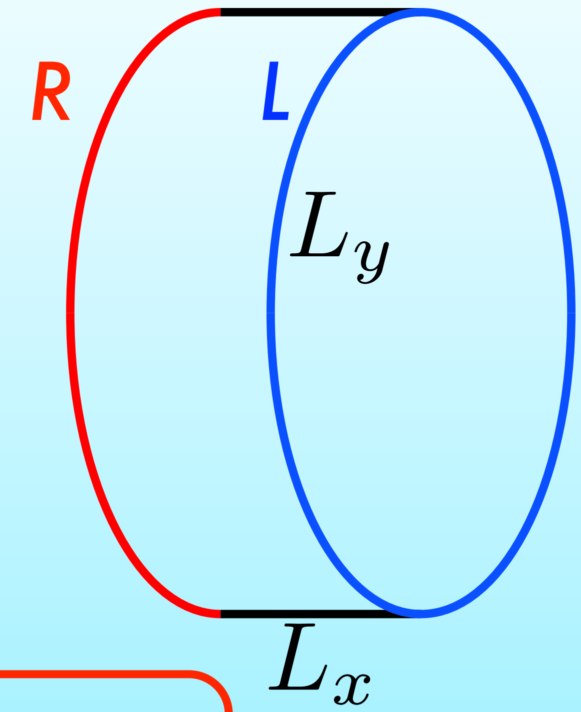




Width (L_x) dependence of the spectrum

☆ *1D system with Boundaries of length* L_x





$$\phi = P/Q$$

$$\boxed{L_x \equiv -1, \quad \mathrm{mod}\, Q}$$

E_B: *independent* of L_x

Q=2q: graphene

Infinite size limit
with Boundaries

Edge State and Bloch State

☆ reduced 1D system and transfer matrix

$$H=\sum_{k_y} H_{1\mathrm{D}}(k_y)$$

Y.H., Phys. Rev. B 48, 11851 (1993)
Phys. Rev. Lett. 71, 3697 (1993)

$$|E,k_y\rangle=\sum_{j_x}\left[\psi_\bullet(E,j_x,k_y)c_\bullet^\dagger(j_x,k_y)|0\rangle+\psi_\circ(E,j_x,k_y)c_\circ^\dagger(j_x,k_y)|0\rangle\right],$$

$$H_{1\mathrm{D}}(k_y)|z,k_y\rangle=z|z,k_y\rangle,\ \ z=E$$

$$M_{\circ\bullet}(j_x)=\begin{pmatrix}\frac{E}{t^*_{\circ\bullet}(j_x)} & -\frac{t_{\bullet\circ}(j_x-1)}{t^*_{\circ\bullet}(j_x)}\\ 1 & 0\end{pmatrix}$$

Transfer matrix $\boldsymbol{\psi}(j_x+1)=M_{\mathrm{t}}(j_x)\boldsymbol{\psi}(j_x)$

$$M_{\bullet\circ}(j_x)=\begin{pmatrix}\frac{E}{t^*_{\bullet\circ}(j_x)} & -\frac{t_{\circ\bullet}(j_x)}{t^*_{\bullet\circ}(j_x)}\\ 1 & 0\end{pmatrix}$$

$$\boldsymbol{\psi}(j_x)=\begin{pmatrix}\psi_\bullet(j_x)\\ \psi_\circ(j_x-1)\end{pmatrix}\qquad M_{\mathrm{t}}(j_x)=M_{\bullet\circ}(j_x)M_{\circ\bullet}(j_x)$$

$$t_{\circ\bullet}(j_x,k_y)=t\left(1+e^{ik_y-i2\pi\phi j_x}\right)$$

$$t_{\bullet\circ}(j_x,k_y)=t\left[1+(t'/t)e^{ik_y-i2\pi\phi(j_x+1/2)}\right]$$

Edge State and Bloch State

☆ reduced 1D system and transfer matrix

$$H = \sum_{k_y} H_{1\mathrm{D}}(k_y)$$

Y.H., Phys. Rev. B 48, 11851 (1993)
Phys. Rev. Lett. 71, 3697 (1993)

$$|E, k_y\rangle = \sum_{j_x} \left[\psi_\bullet(E, j_x, k_y) c_\bullet^\dagger(j_x, k_y)|0\rangle + \psi_\circ(E, j_x, k_y) c_\circ^\dagger(j_x, k_y)|0\rangle \right],$$

$$H_{1\mathrm{D}}(k_y)|z, k_y\rangle = z|z, k_y\rangle, \;\; z = E$$

$$M_{\circ\bullet}(j_x) = \begin{pmatrix} \frac{E}{t^*_{\circ\bullet}(j_x)} & -\frac{t_{\bullet\circ}(j_x-1)}{t^*_{\circ\bullet}(j_x)} \\ 1 & 0 \end{pmatrix}$$

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Transfer matrix $\boldsymbol{\psi}(j_x + 1) = M_{\mathrm{t}}(j_x)\boldsymbol{\psi}(j_x)$

$$t_{\circ\bullet}(j_x, k_y) = t\left(1 + e^{ik_y - i2\pi\phi j_x}\right)$$

$$t_{\bullet\circ}(j_x, k_y) = t\left[1 + (t'/t)e^{ik_y - i2\pi\phi(j_x+1/2)}\right]$$

$$\boldsymbol{\psi}(j_x) = \begin{pmatrix} \psi_\bullet(j_x) \\ \psi_\circ(j_x - 1) \end{pmatrix} \qquad M_{\mathrm{t}}(j_x) = M_{\bullet\circ}(j_x)M_{\circ\bullet}(j_x)$$

Boundary Conditions

Edge State and Bloch State

☆ reduced 1D system and transfer matrix

$$H = \sum_{k_y} H_{1\mathrm{D}}(k_y)$$

Y.H., Phys. Rev. B 48, 11851 (1993)
Phys. Rev. Lett. 71, 3697 (1993)

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<u>Boundary Conditions</u>

Bloch State

$$\boldsymbol{\psi}_B(q) = M\boldsymbol{\psi}_B(0) = \rho\boldsymbol{\psi}_B(0)$$

$$|\rho| = 1$$

$$M = M_{\mathrm{t}}(q-1) M_{\mathrm{t}}(q-2) \cdots M_{\mathrm{t}}(0)$$

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Boundary Conditions

Bloch State

$$\boldsymbol{\psi}_B(q)=M\boldsymbol{\psi}_B(0)=\rho\boldsymbol{\psi}_B(0)$$

$$|\rho|=1$$

Edge State

$$\boldsymbol{\psi}_E(0)=\begin{pmatrix}1\\0\end{pmatrix},\quad \boldsymbol{\psi}_E(q)=\begin{pmatrix}*\\0\end{pmatrix}$$

$$M=M_{\mathrm{t}}(q-1)M_{\mathrm{t}}(q-2)\cdots M_{\mathrm{t}}(0)$$

Edge State and Bloch State

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$$\boldsymbol{\psi}(j_x)=\begin{pmatrix}\psi_\bullet(j_x)\\ \psi_\circ(j_x-1)\end{pmatrix}\qquad M_{\mathrm{t}}(j_x)=M_{\bullet\circ}(j_x)M_{\circ\bullet}(j_x)$$

How these two are related ??

Bloch State

$$\boldsymbol{\psi}_B(q)=M\boldsymbol{\psi}_B(0)=\rho\boldsymbol{\psi}_B(0)$$

$$|\rho|=1$$

Edge State

$$\boldsymbol{\psi}_E(0)=\begin{pmatrix}1\\0\end{pmatrix},\quad \boldsymbol{\psi}_E(q)=\begin{pmatrix}*\\0\end{pmatrix}$$

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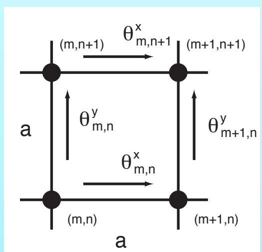
Hofstadter Problem & Graphene under magnetic field

Topol. char. by edges

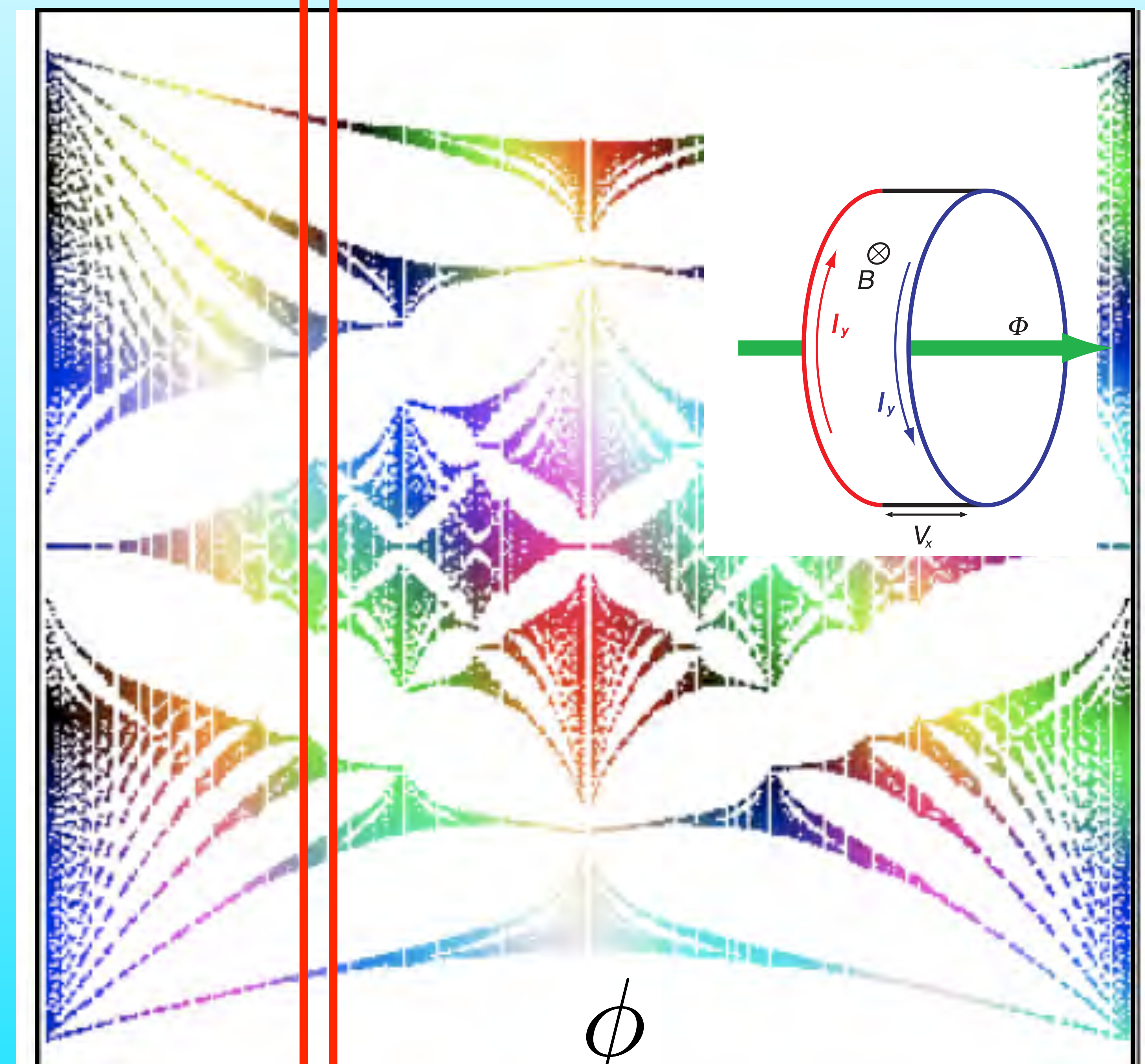
☆ In continuum, 2D = $\sum_{k_y}$(1D harmonic oscillators with parameter k_y)

Landau gauge

☆ Bloch electrons, 2D =$\sum_{k_y}$(1D Harper equation with parameter k_y)



Energy



Edge State and Bloch State

Topol. char. by edges

- Bloch electrons, 2D = $\sum$(1D Harper problem with parameter k_y)

As for the 1D Harper equation,

- Edge state : bound state
- Bloch state: scattering state

These two can be treated in a unified way by considering complex energy

standard quantum mechanics

- bound state
- scattering state

$$E = \frac{\hbar^2 k^2}{2m} \begin{cases} < 0 & k = i\kappa, \quad \psi \sim e^{-\kappa x} \\ > 0 & k \in \mathbb{R}, \quad \psi \sim e^{ikx} \end{cases}$$

E = z (complex energy)

branch cut

$z = E - i0$

$E > 0$

$E < 0$

unified description

$\psi \sim e^{i\sqrt{2mE}\, x/\hbar}$

energy of the bound state is in the gap region E<0

Hofstadter Problem & Graphene under magnetic field

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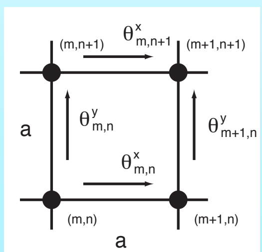
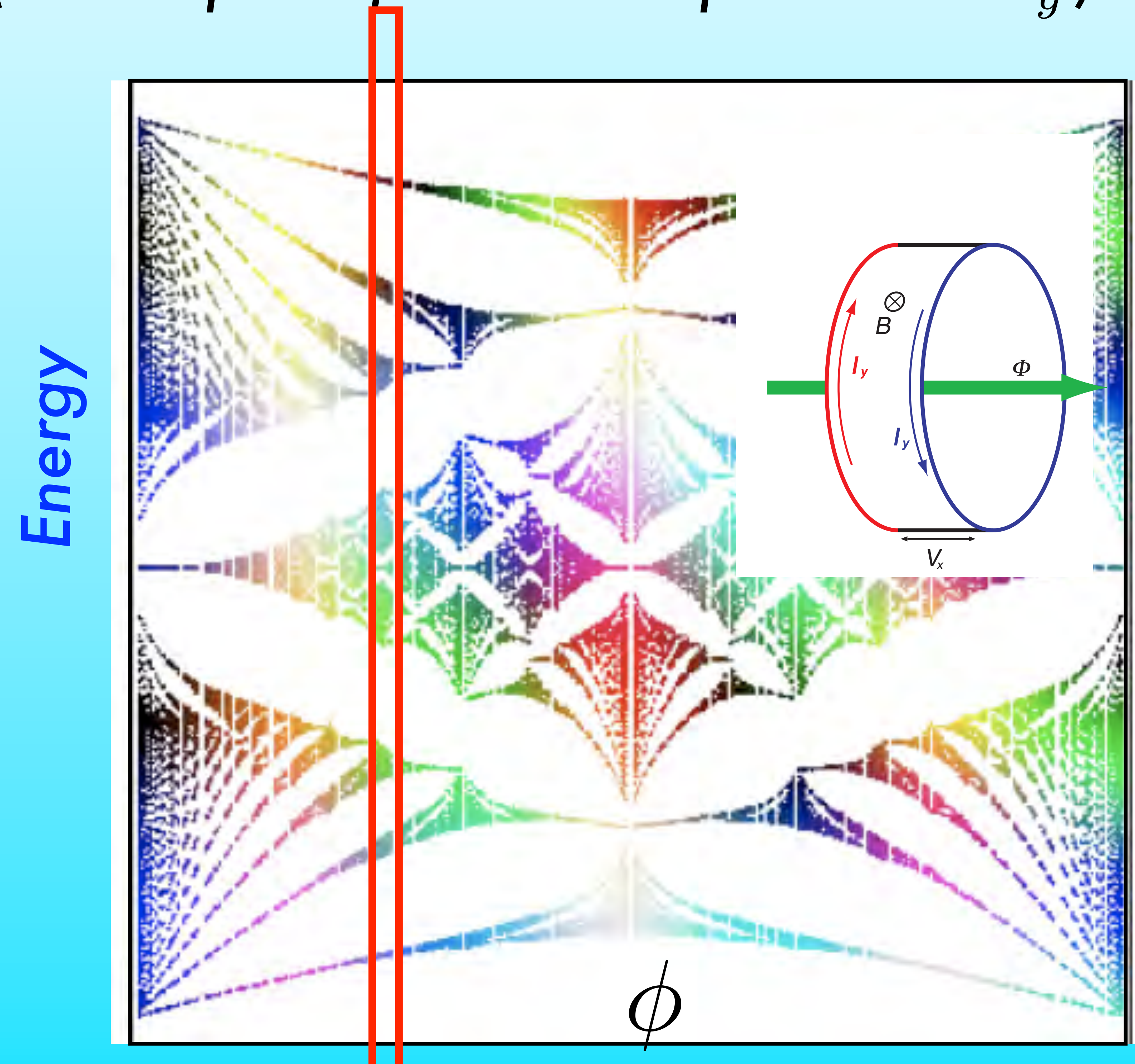


Complex energy surface
of
the Harper eq.



Hofstadter Problem & Graphene under magnetic field

Topol. char. by edges

☆ *In continuum, 2D =* $\sum_{k_y}$ *(1D harmonic oscillators with parameter* k_y *)*

Landau gauge

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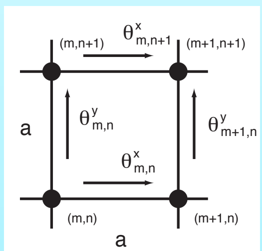

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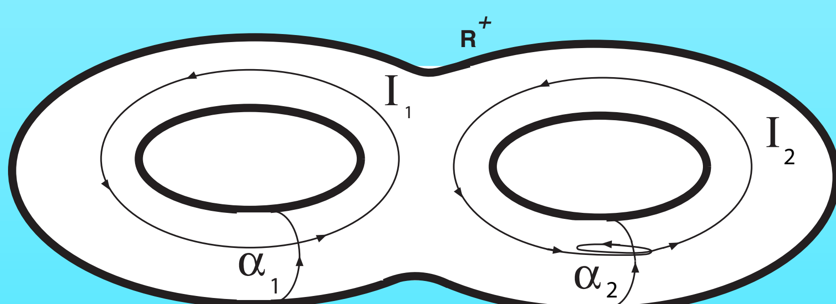
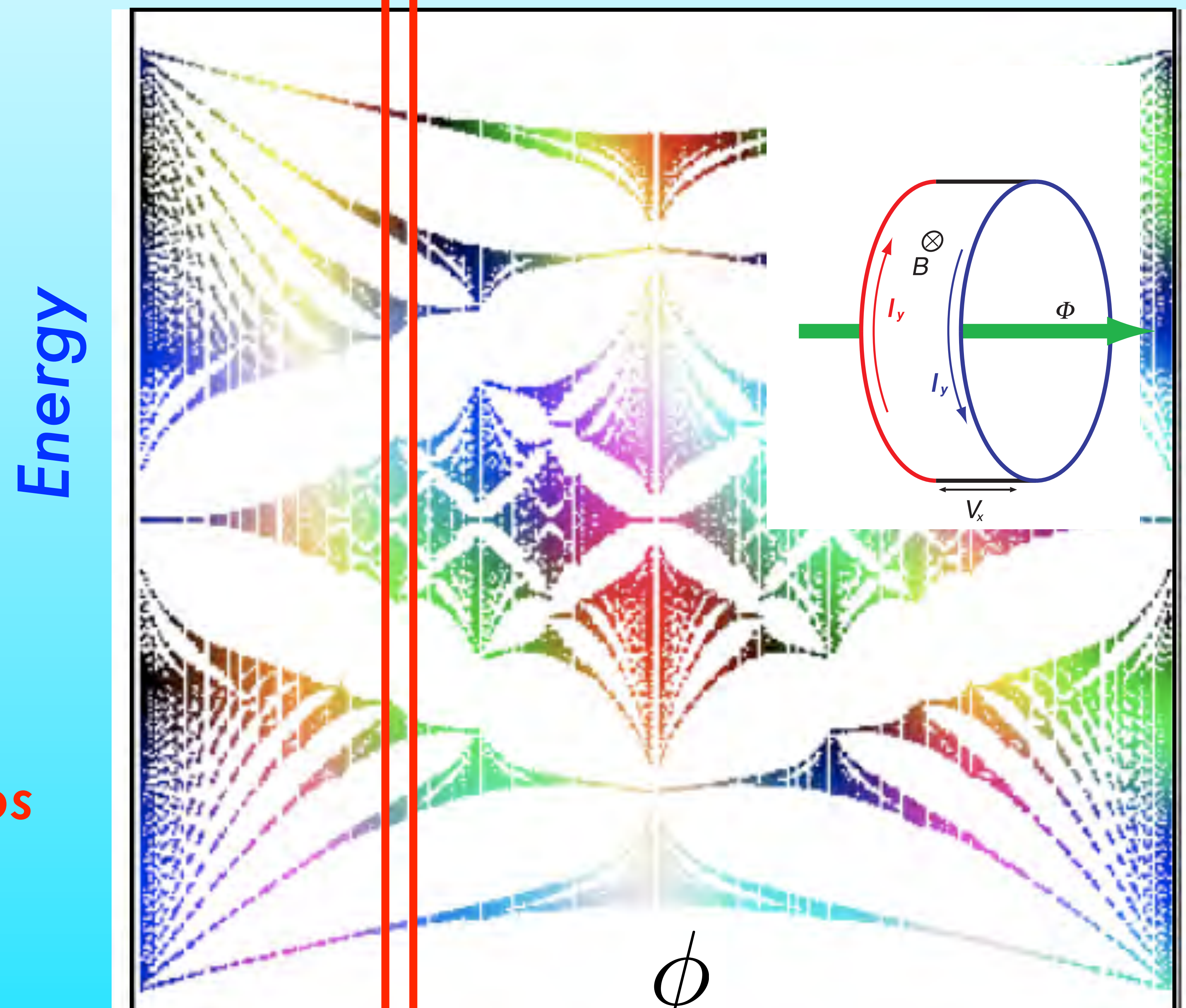


q Bands and g=q-1 gaps

Riemann surface

with g handles



Edge states are topological

Quantized Hall conductance by the topological number of edge states

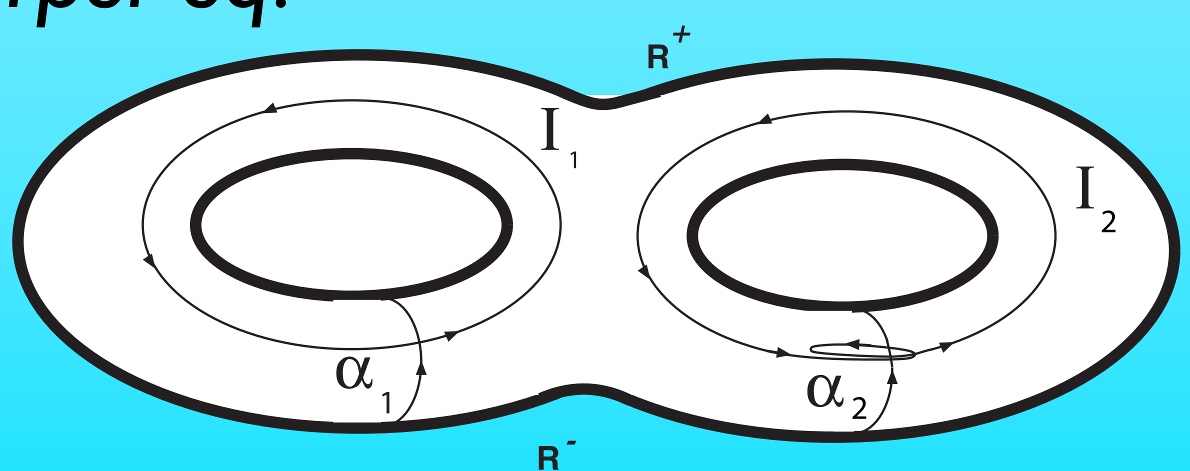
$$\sigma_{xy}{}^{\text{edge}} = \frac{e^2}{h} I_j$$

Topological number

I_j : Winding # of the edge state energy around the handle (energy gap) on the complex energy surface

Complex Energy surface of Harper eq.

Y. Hatsugai, Phys. Rev. B 48, 11851–11862 (1993)



genus g=q – 1: number of the gaps

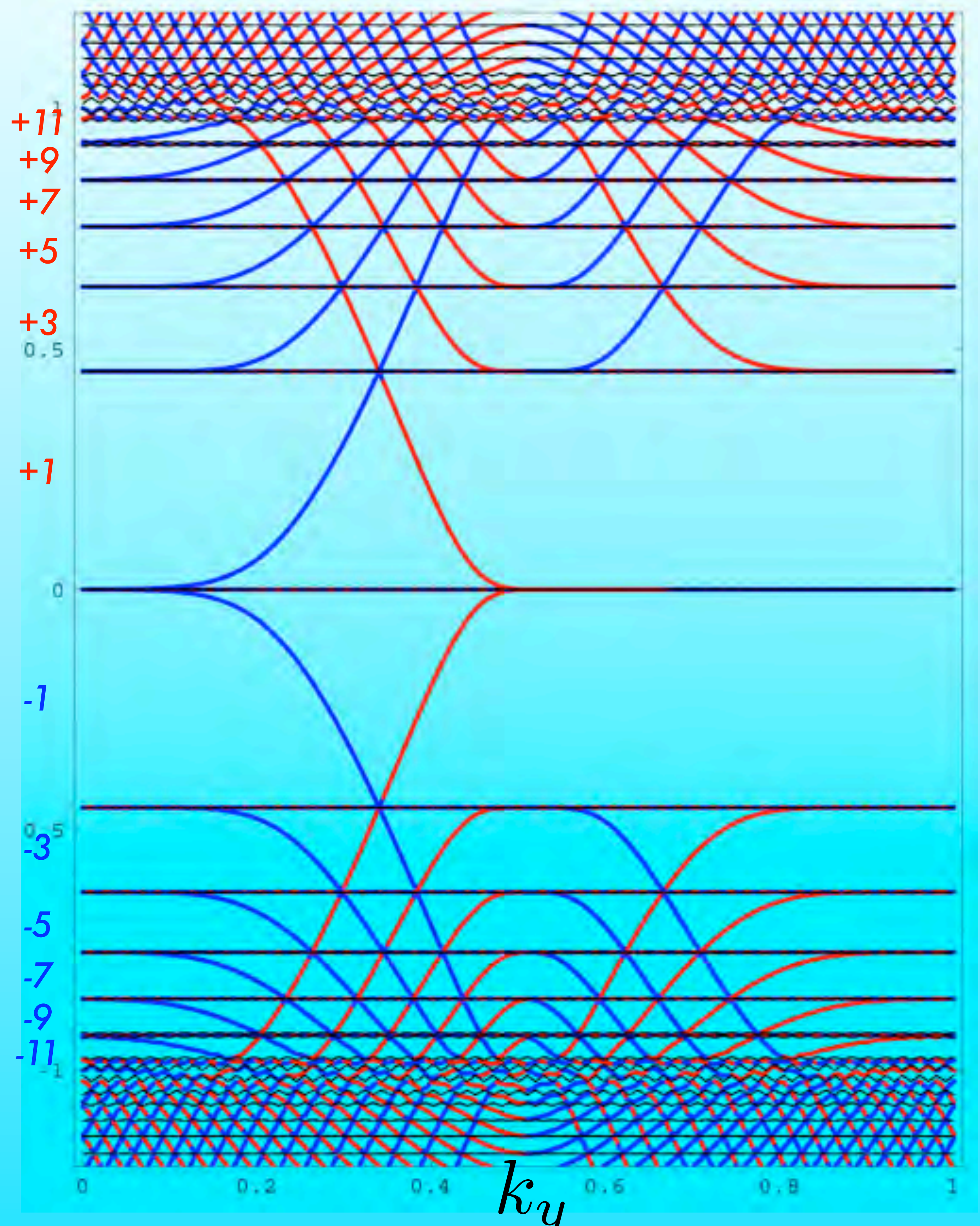
$$\phi = p/q$$

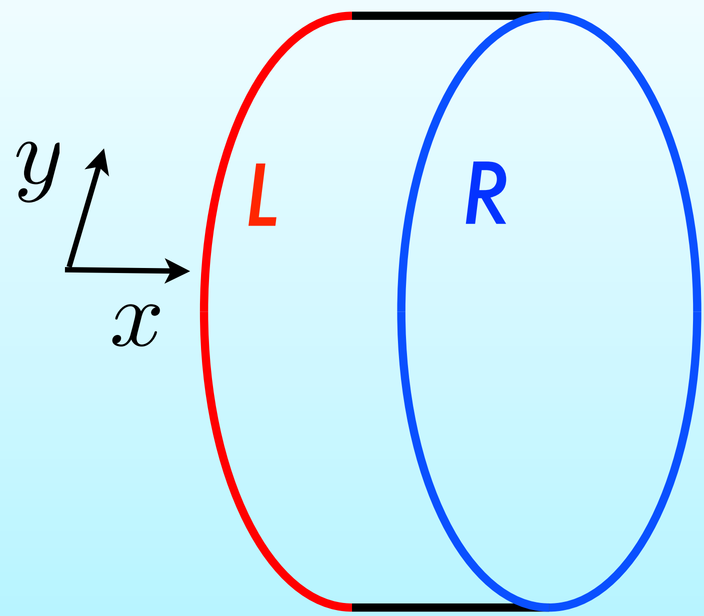
Edge States of Graphene

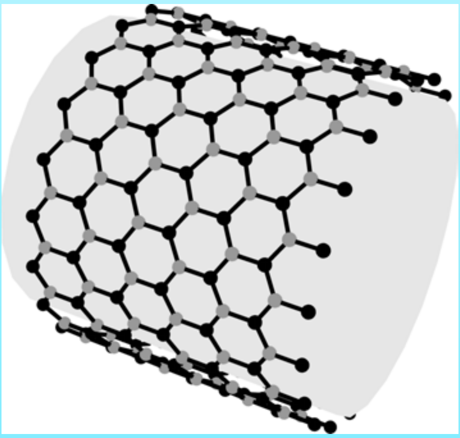
Topol. char. by edges

$\phi = 1/51$

Edge States being consistent with Dirac Type Quantization







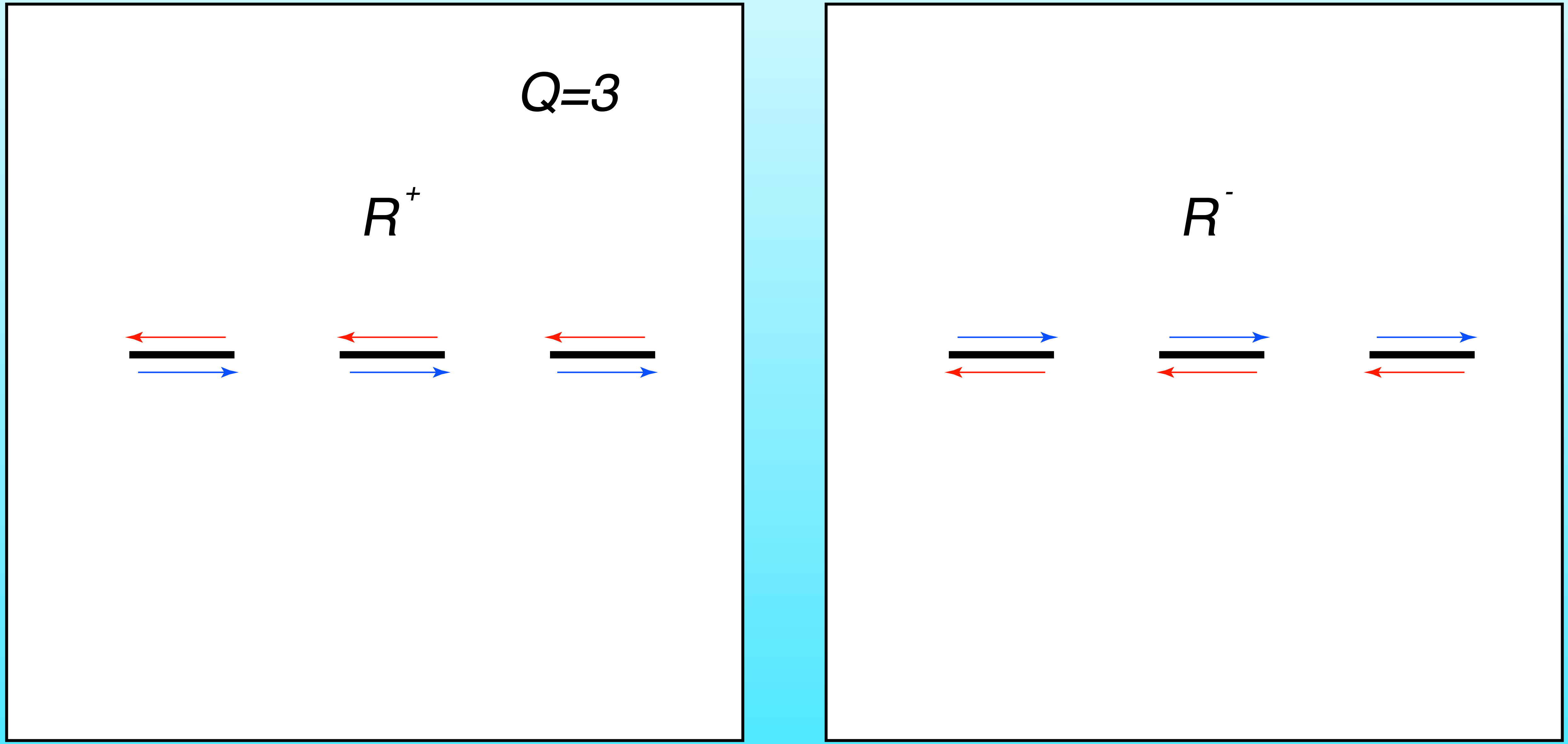
Construction of the Riemann surface

$\phi = 1/3$

$$\sqrt{(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_{2Q-1})(z-\lambda_{2Q})}$$

- Glue 2 complex planes

Q=3 energy bands: Q=3 branch cuts



Construction of the Riemann surface

$\Phi = P/Q, \quad Q = 3$

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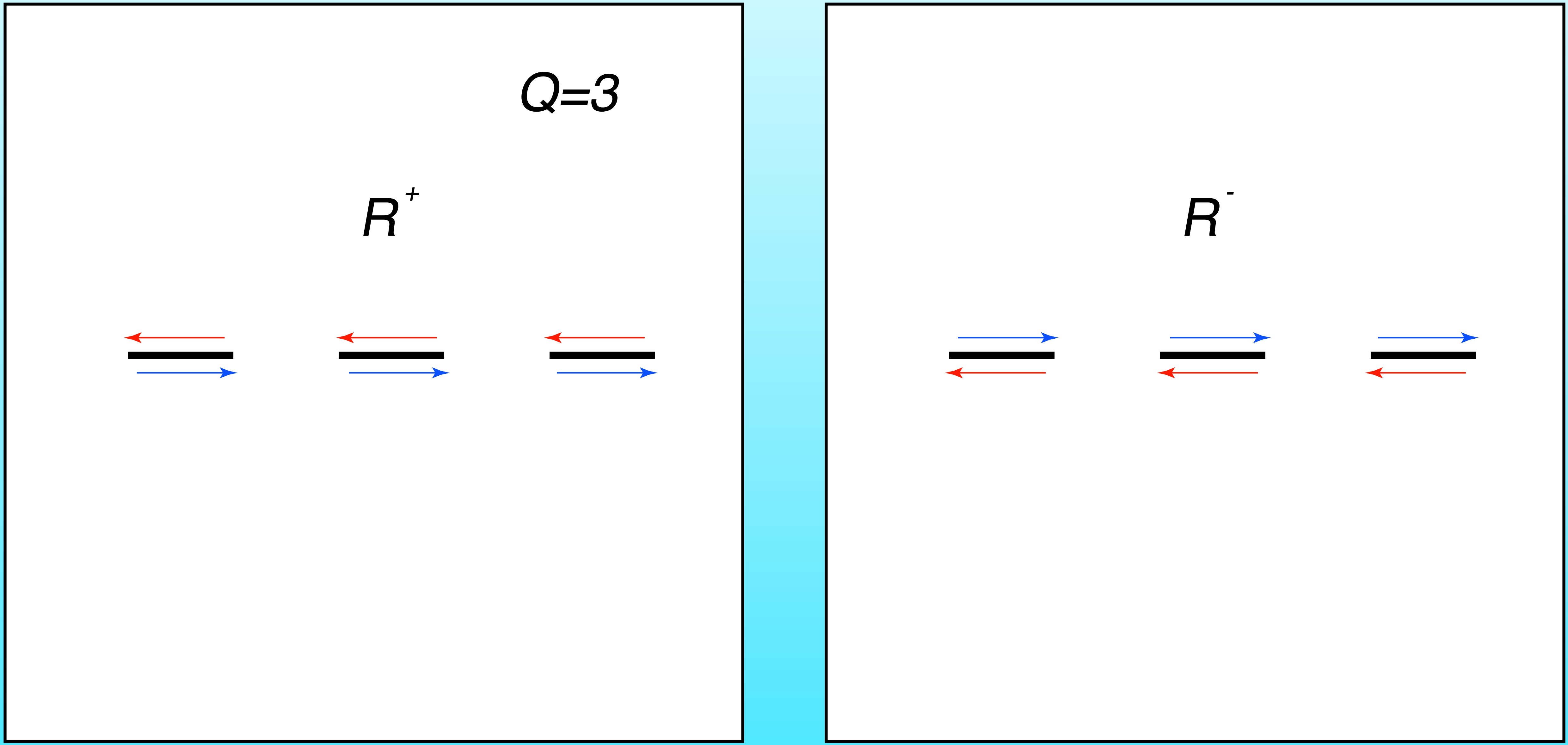
- Glue 2 complex planes with Q branch cuts

Q=3 energy bands: Q=3 branch cuts



g=Q-1 holes

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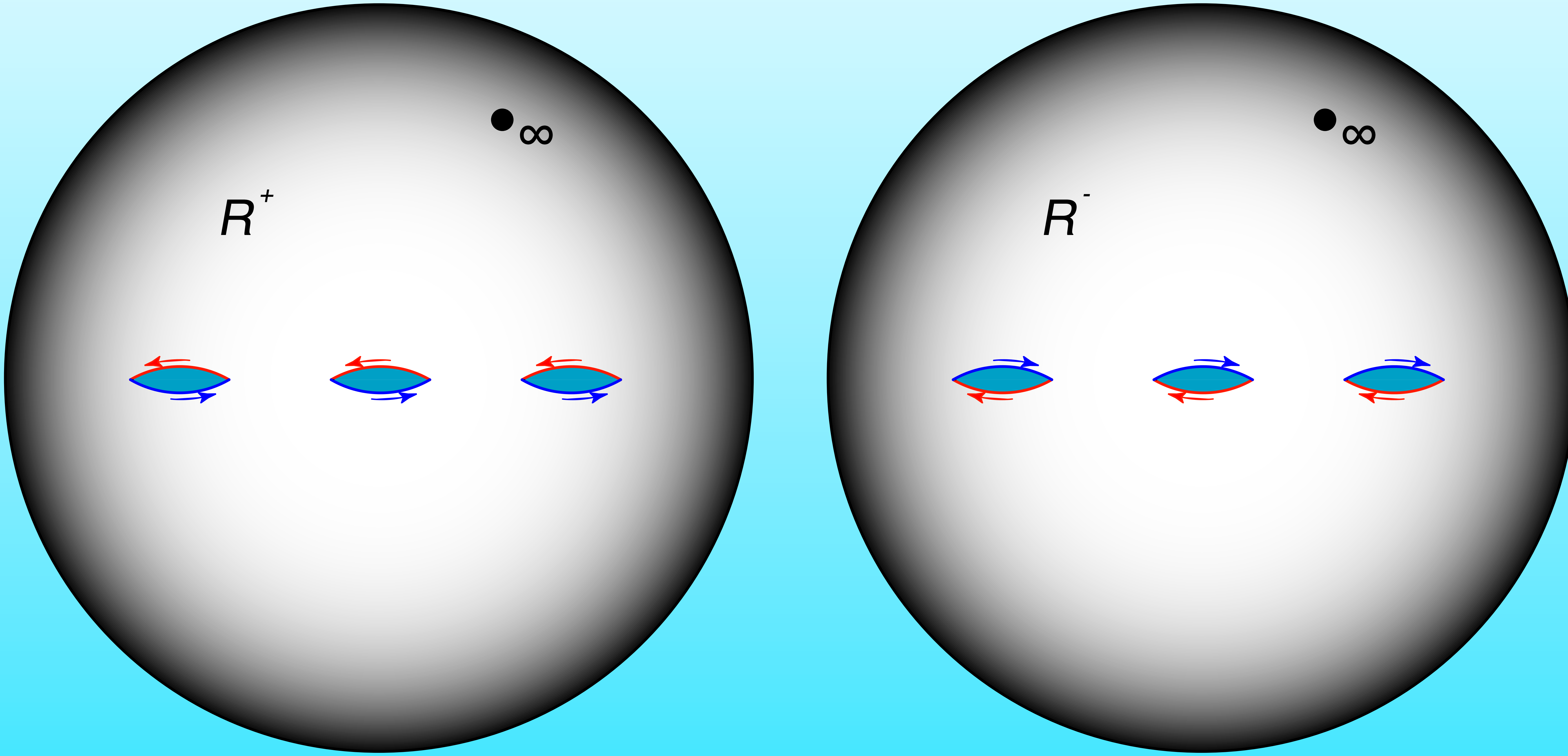

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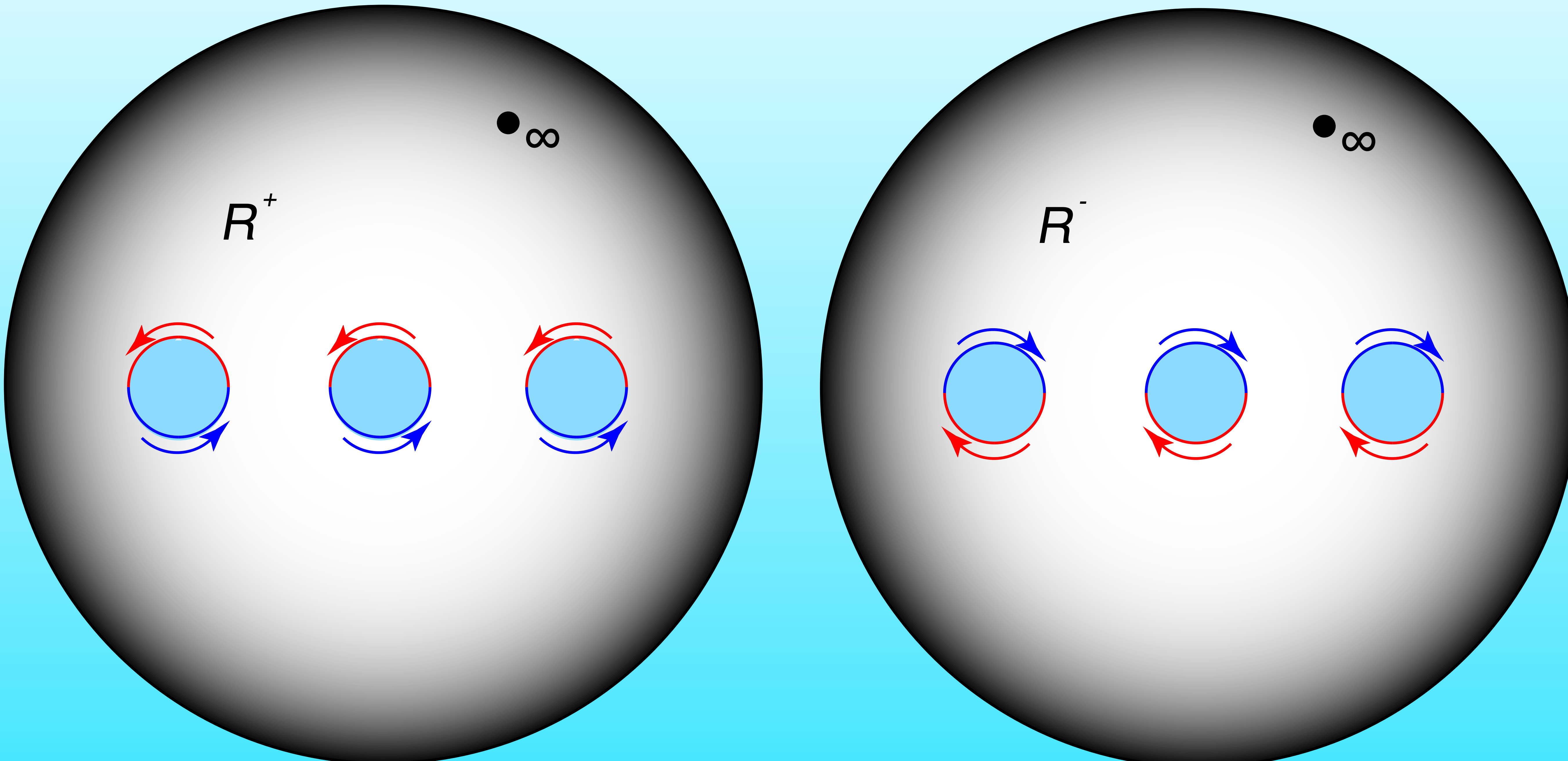

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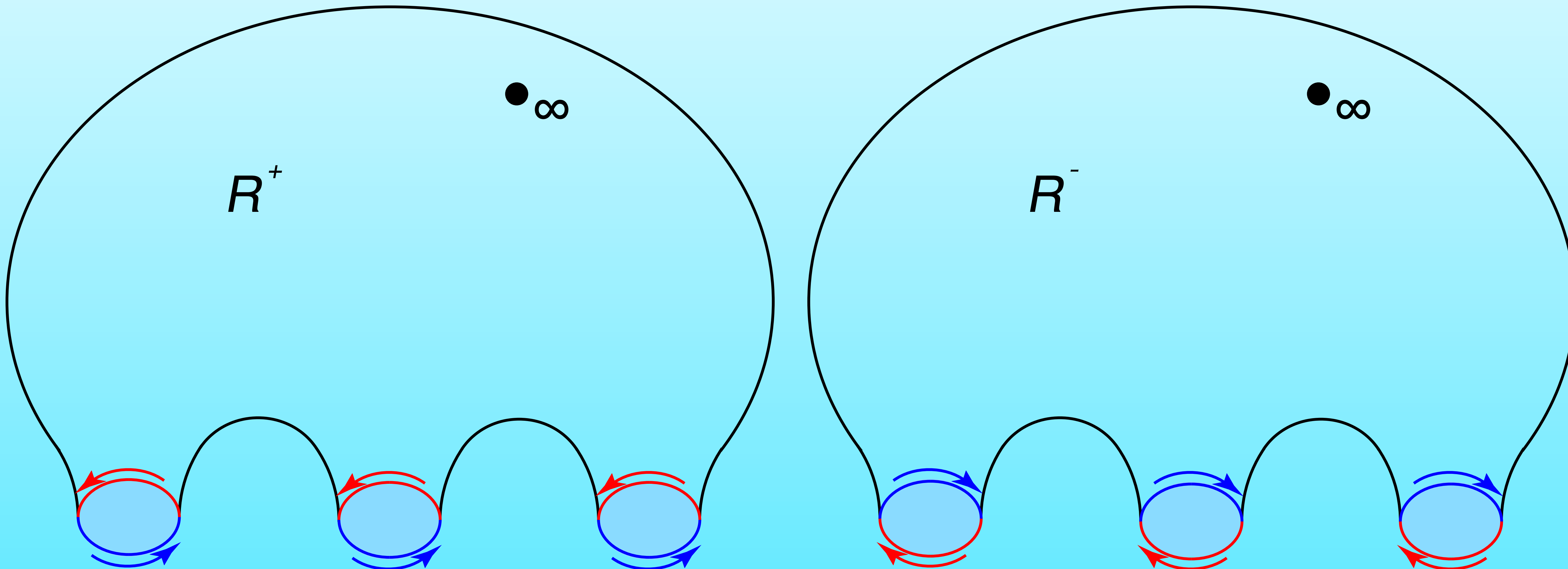

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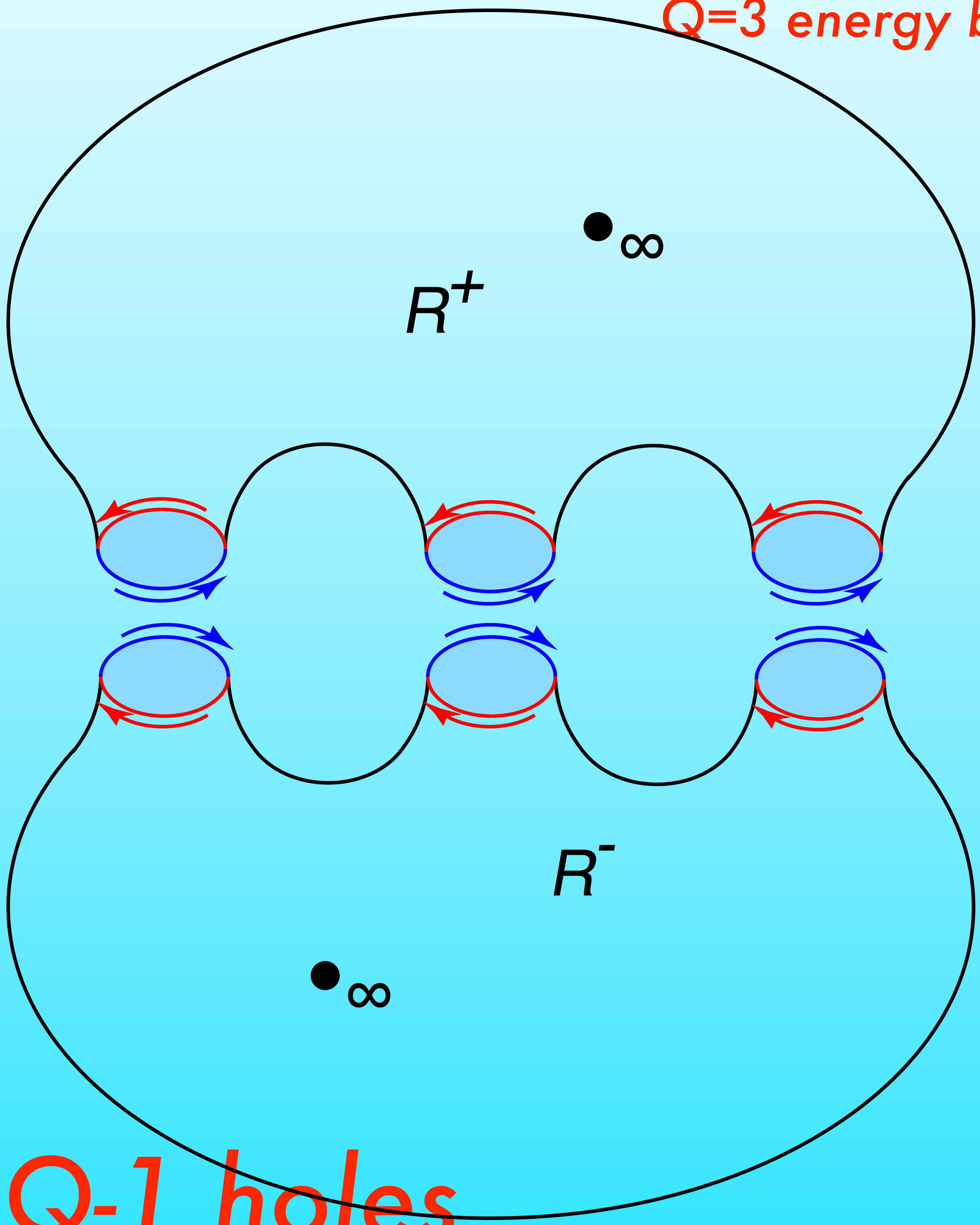

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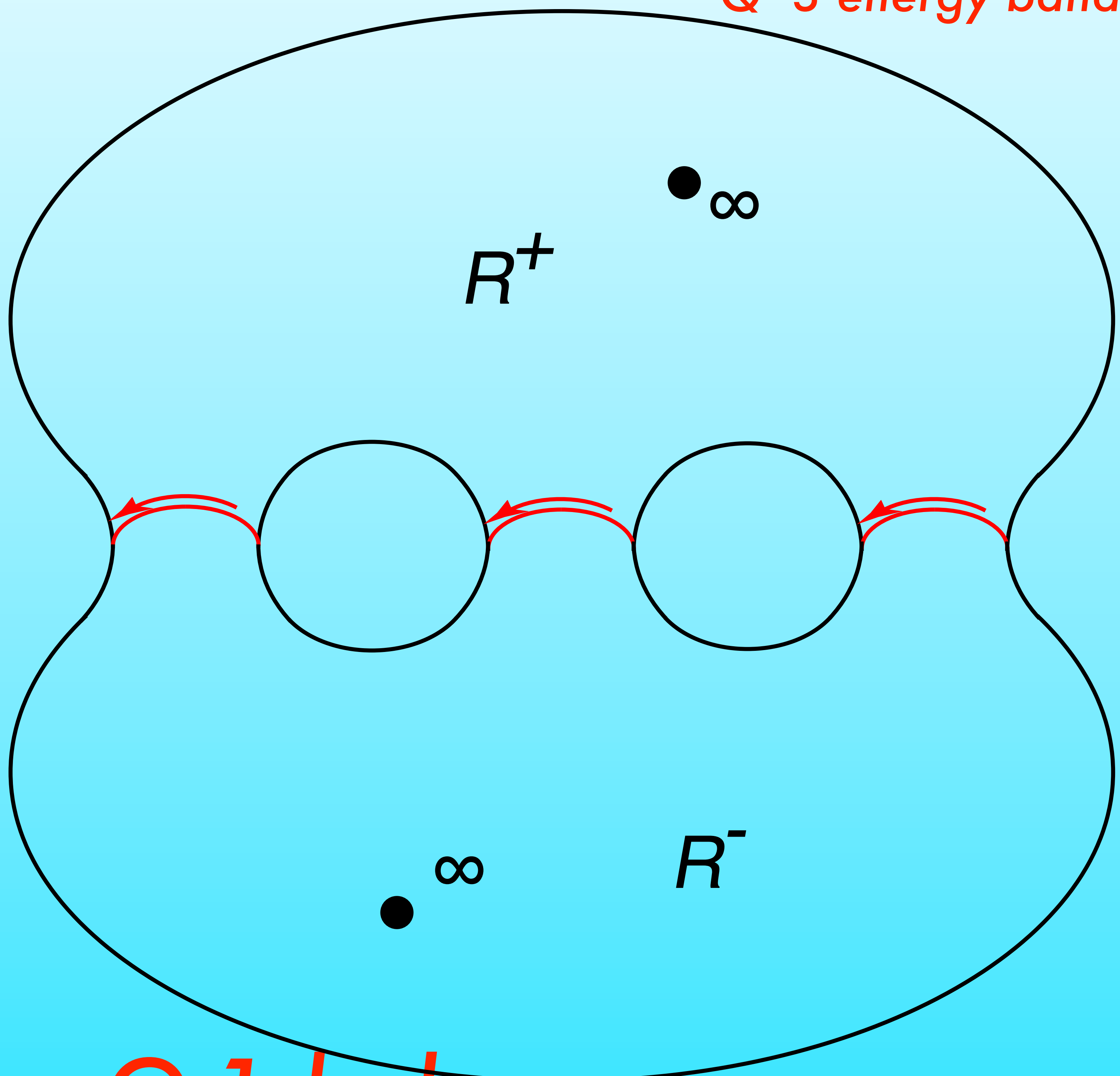

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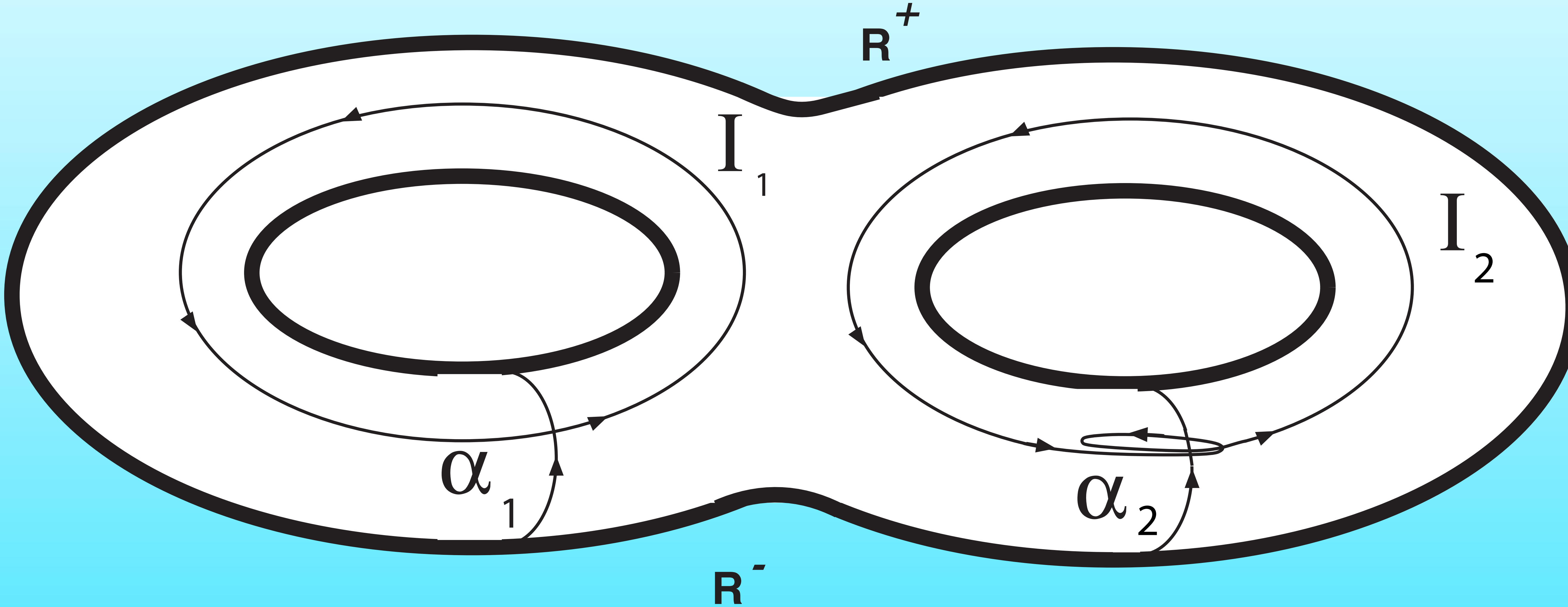

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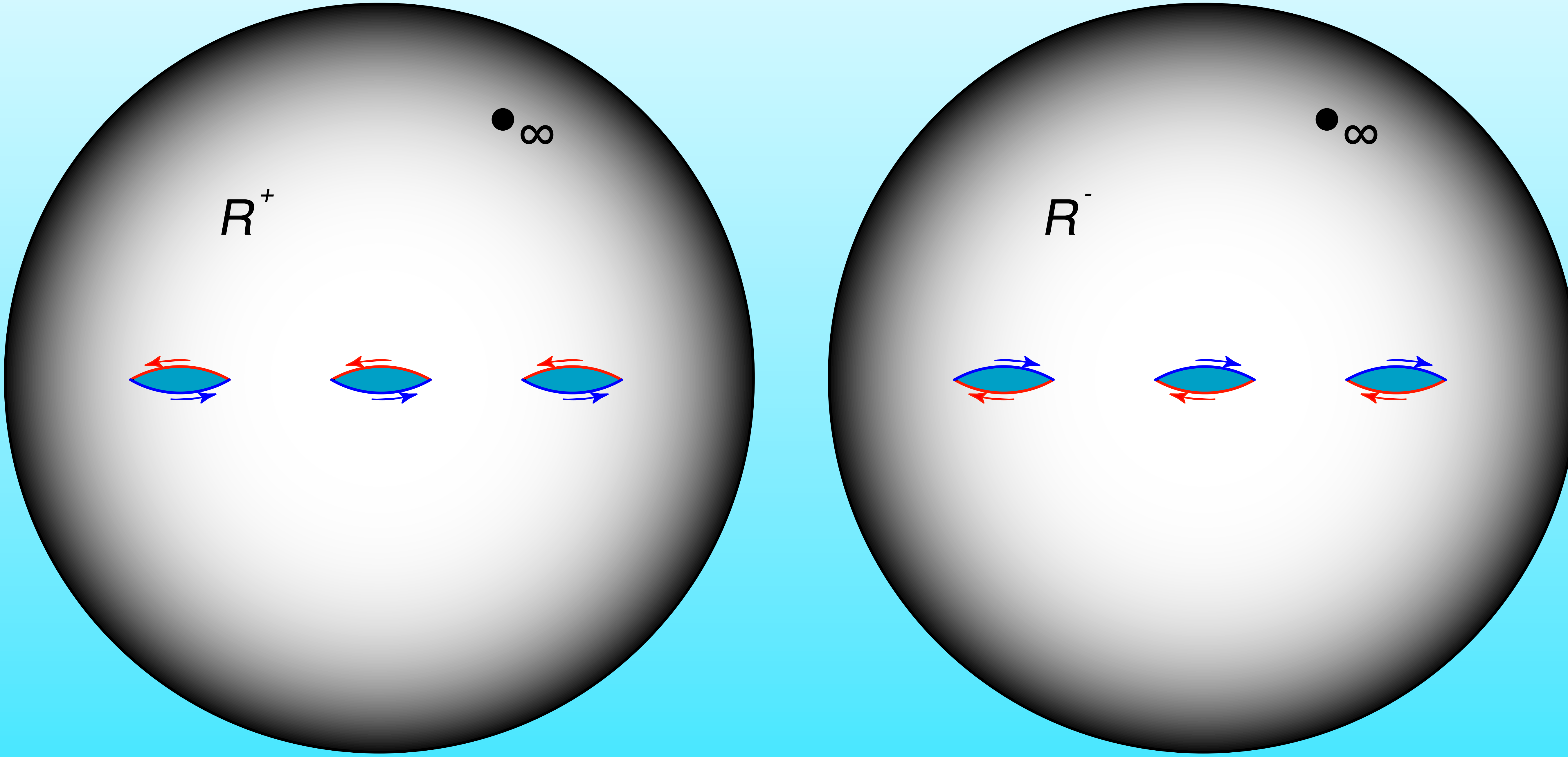

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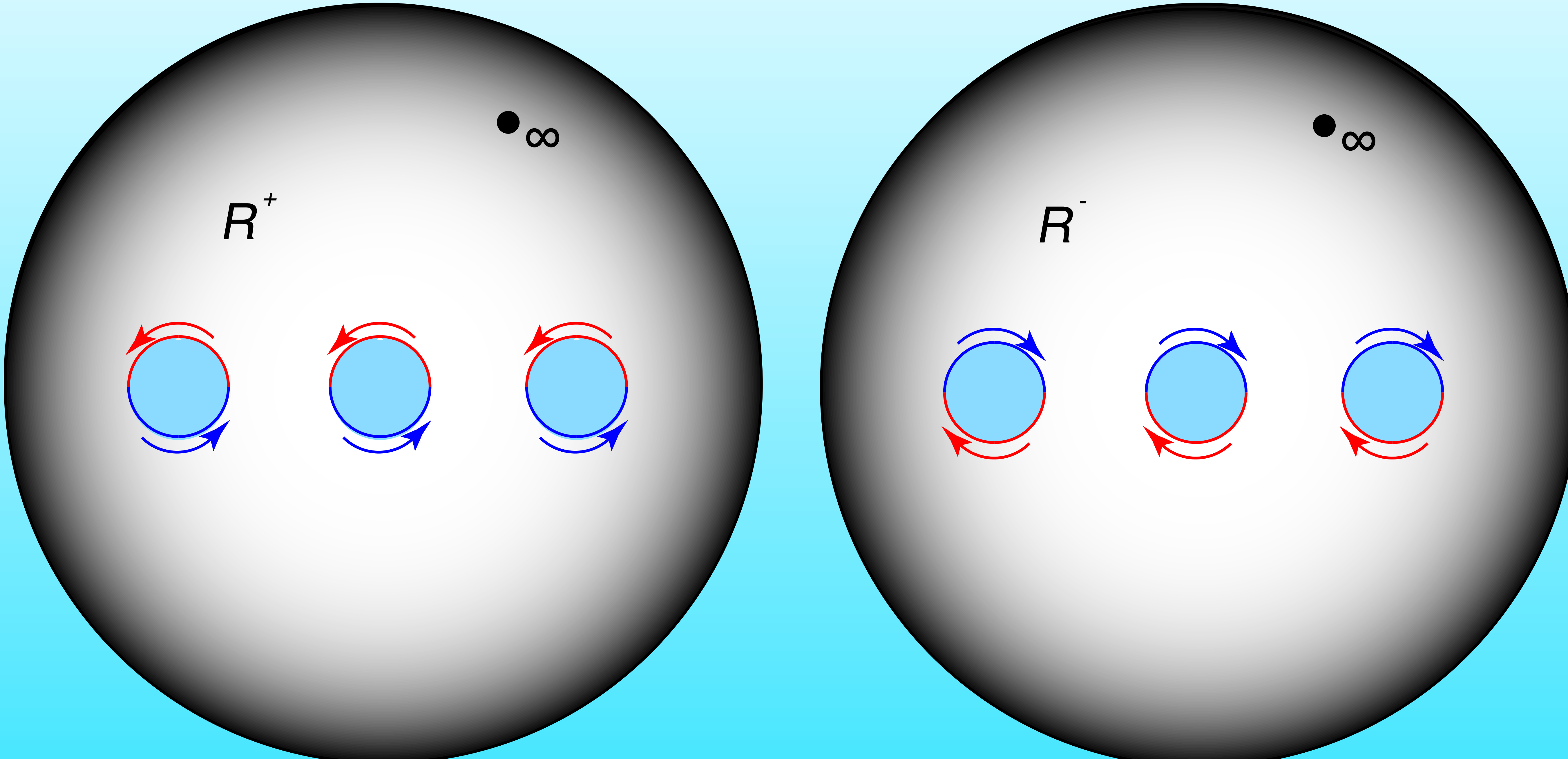

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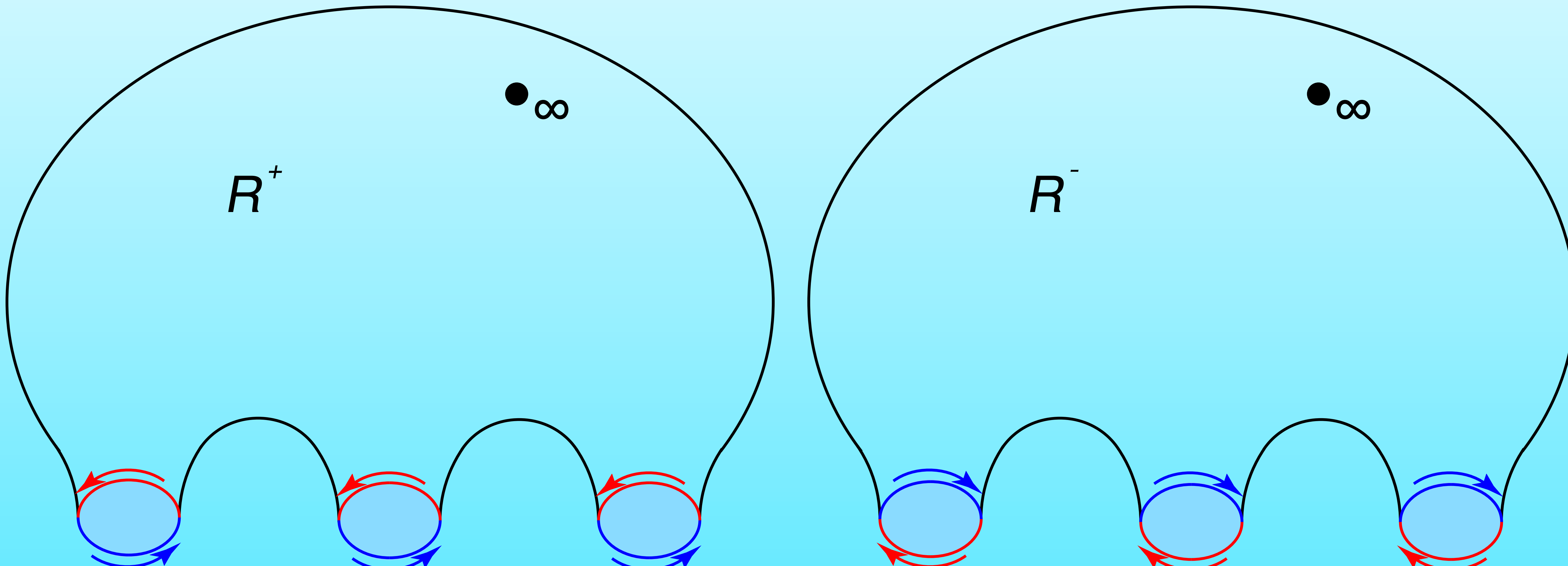

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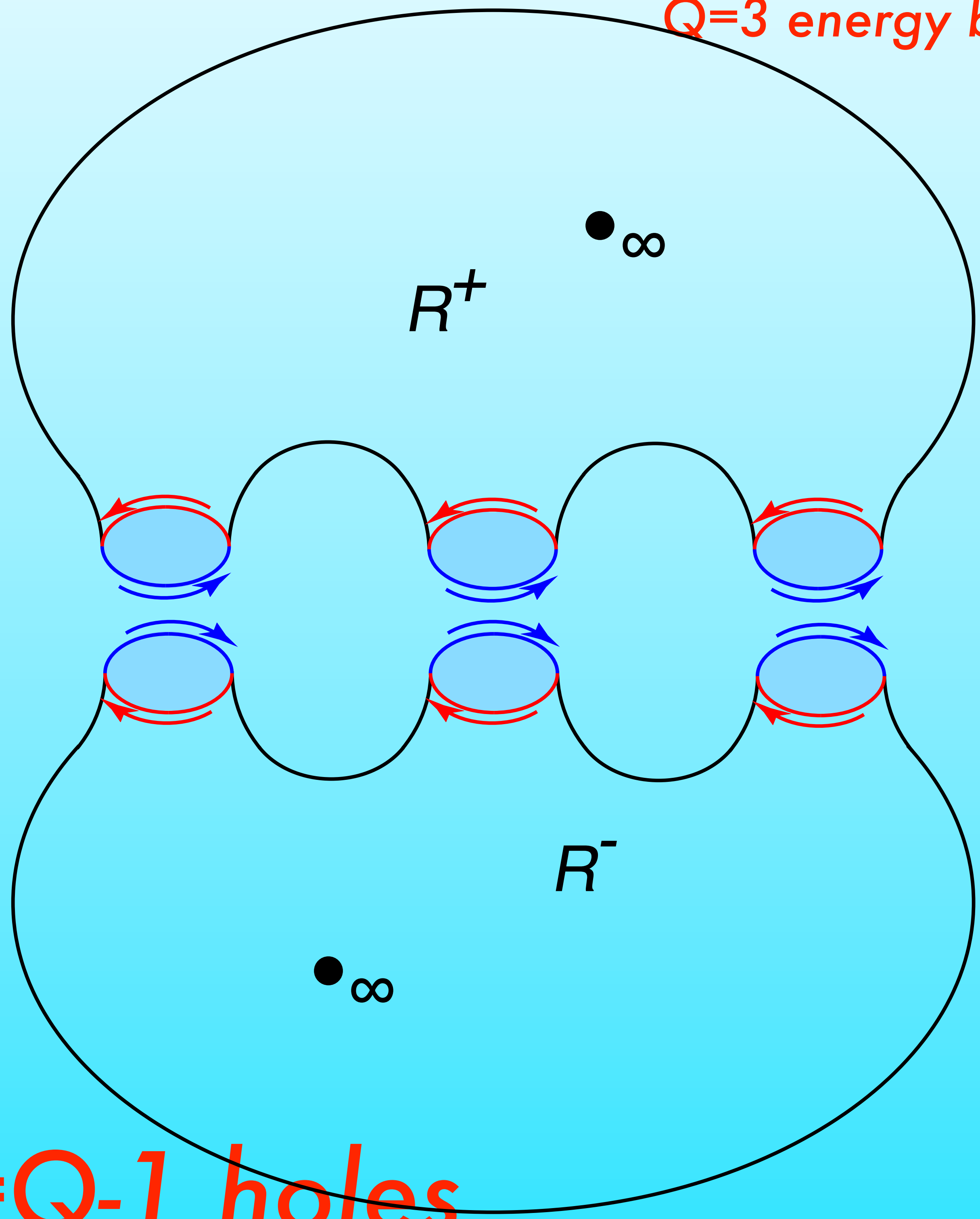

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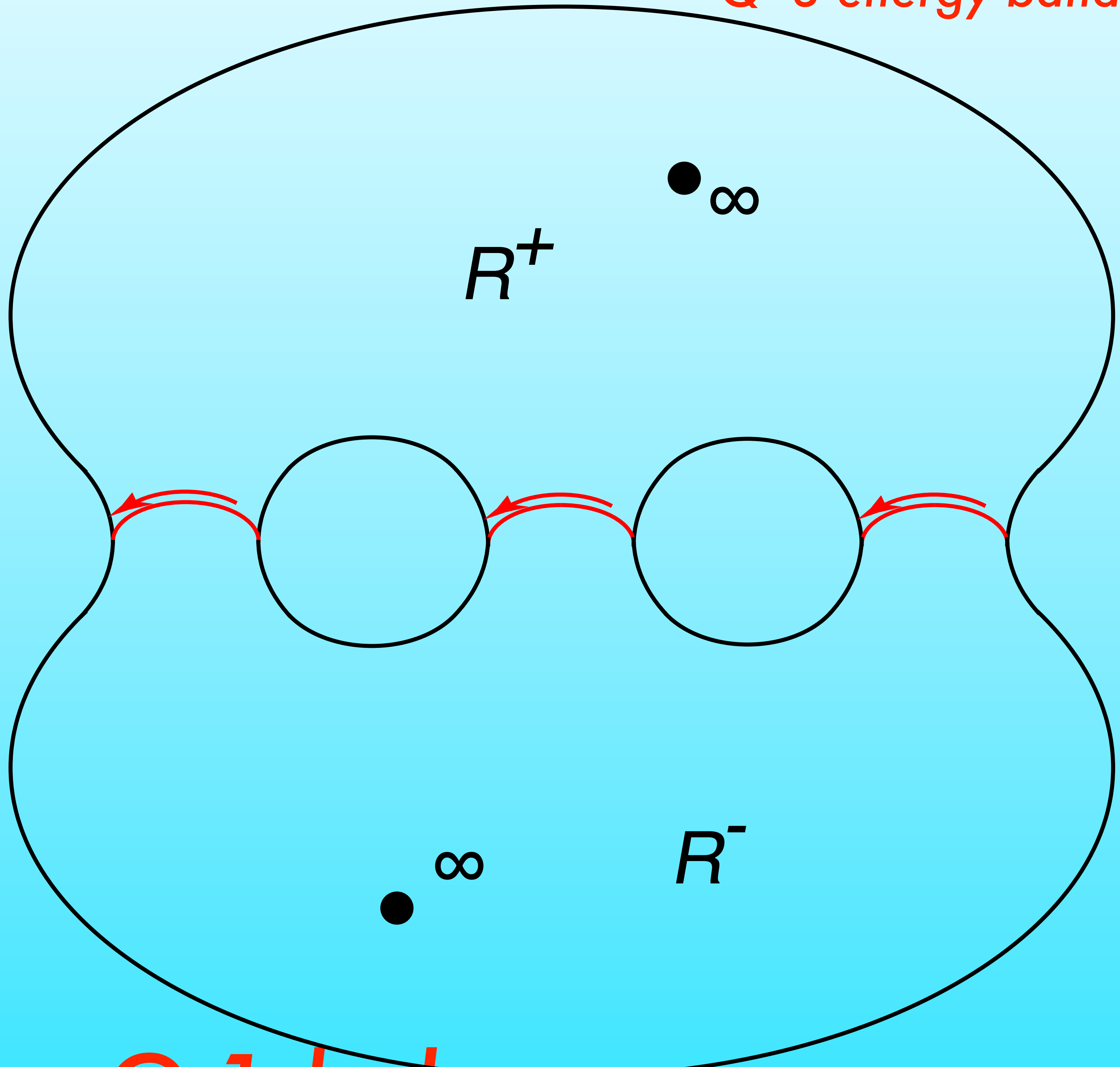

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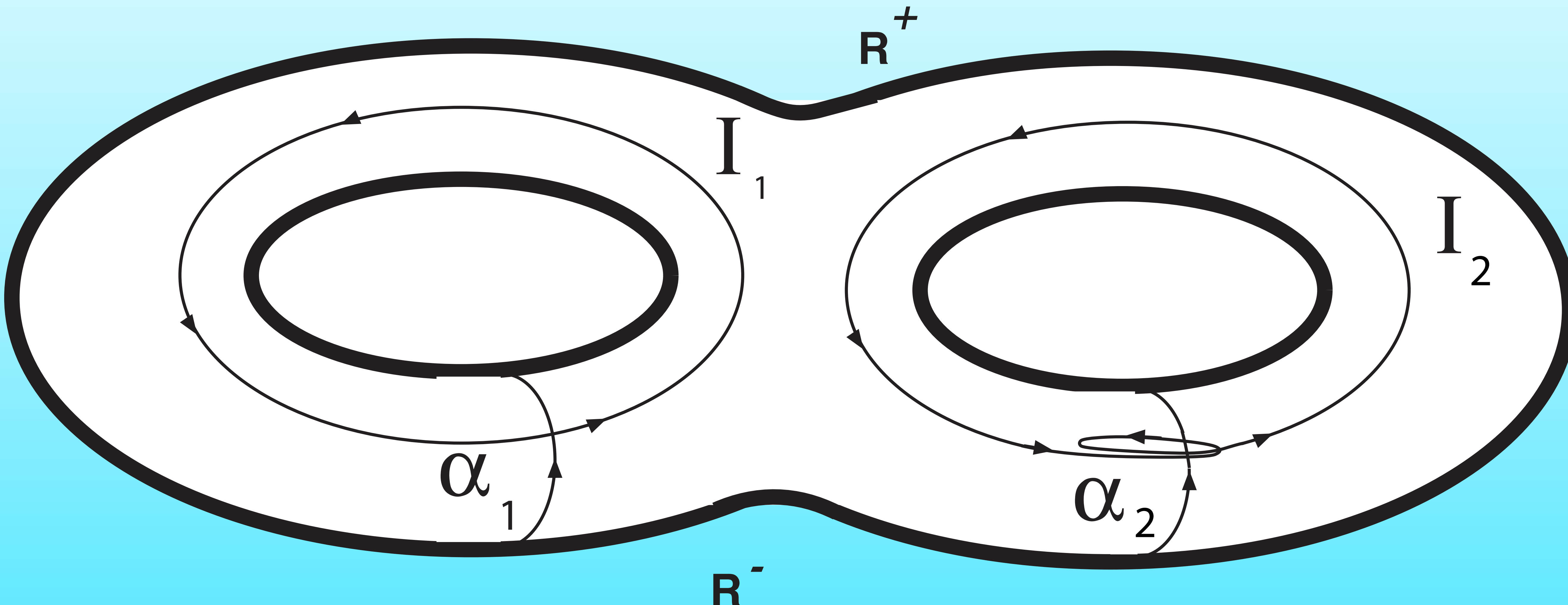

g=Q-1 holes

Wave function & Riemann Surface Σ_g

As for fixed k_y of the 1D-Harper systems

☆ Zeros of the Bloch fn. defines the Edge State Energies

$$\phi = P/Q$$
$$g = Q - 1$$

Energy bands ⟷ Branch cuts

Energy gaps ⟷ Holes

W. fn. is localized at ... The zero of the Bloch fn. is on

the left edge ⟷ the upper Riemann Surface R^+

the right edge ⟷ the upper Riemann Surface R^-

☆ Changing $k_y \in [0, 2\pi]$, the zero of the Bloch function in the j-th gap makes a closed loop on Σ_g

Wave function & Riemann Surface Σ_g

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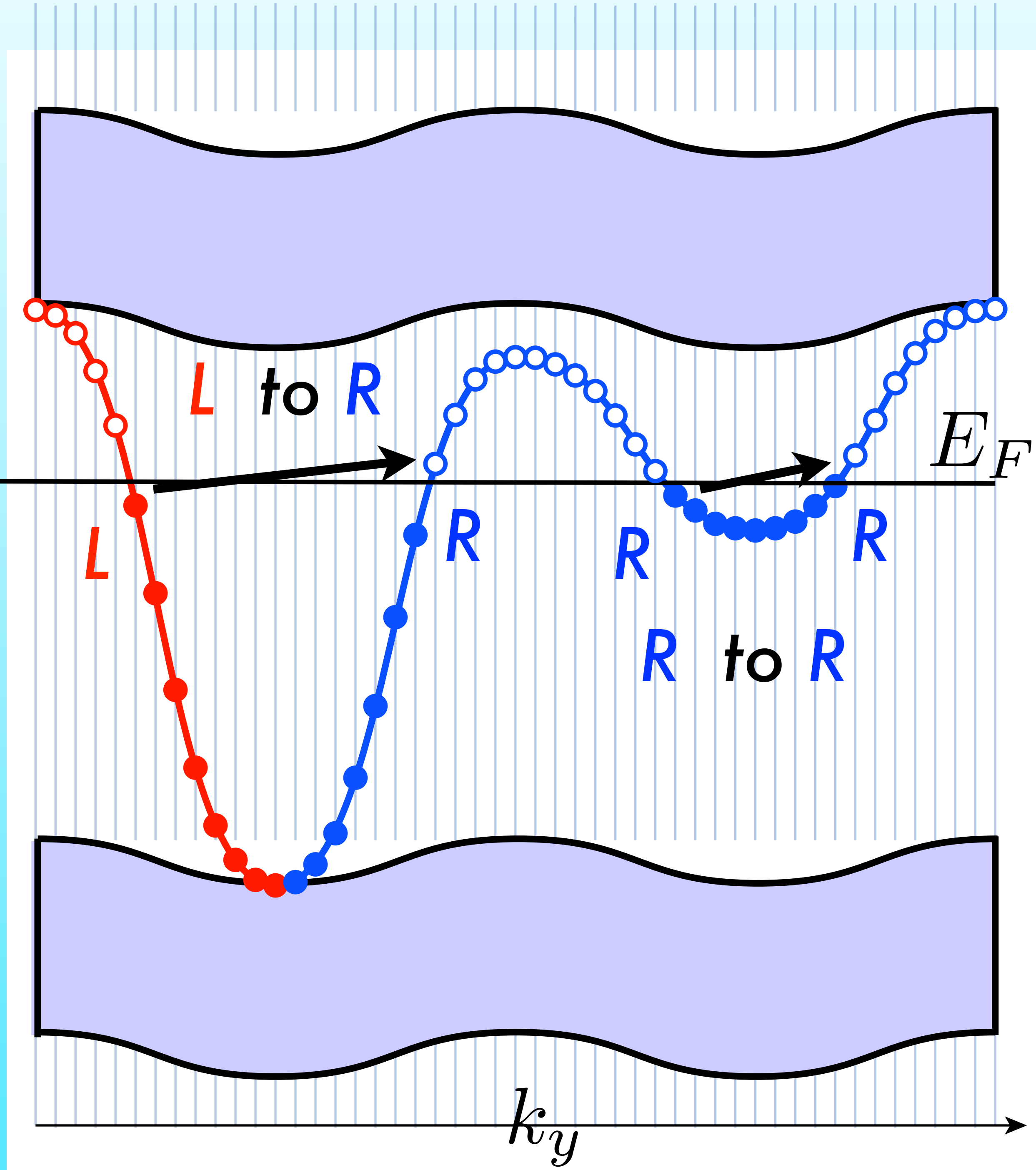
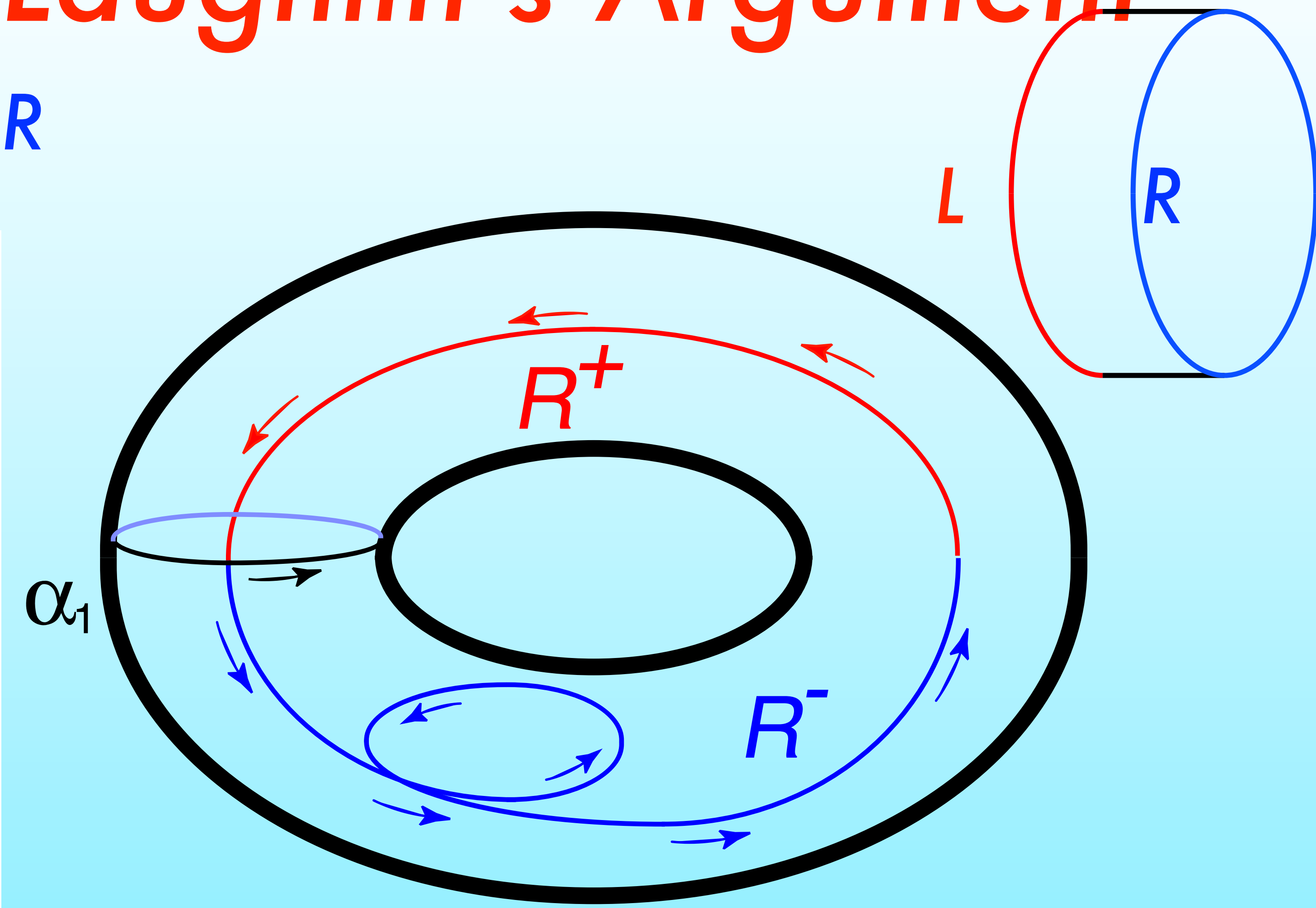
Riemann surface & Laughlin's Argument

1 state is carried from the L to R





$$I(\alpha_j, L^j_{\rm edge}) = +1, \; j=1$$

Topological number

Y.H., Phys. Rev. B 48, 11851 (1993)

Riemann surface & Laughlin's Argument

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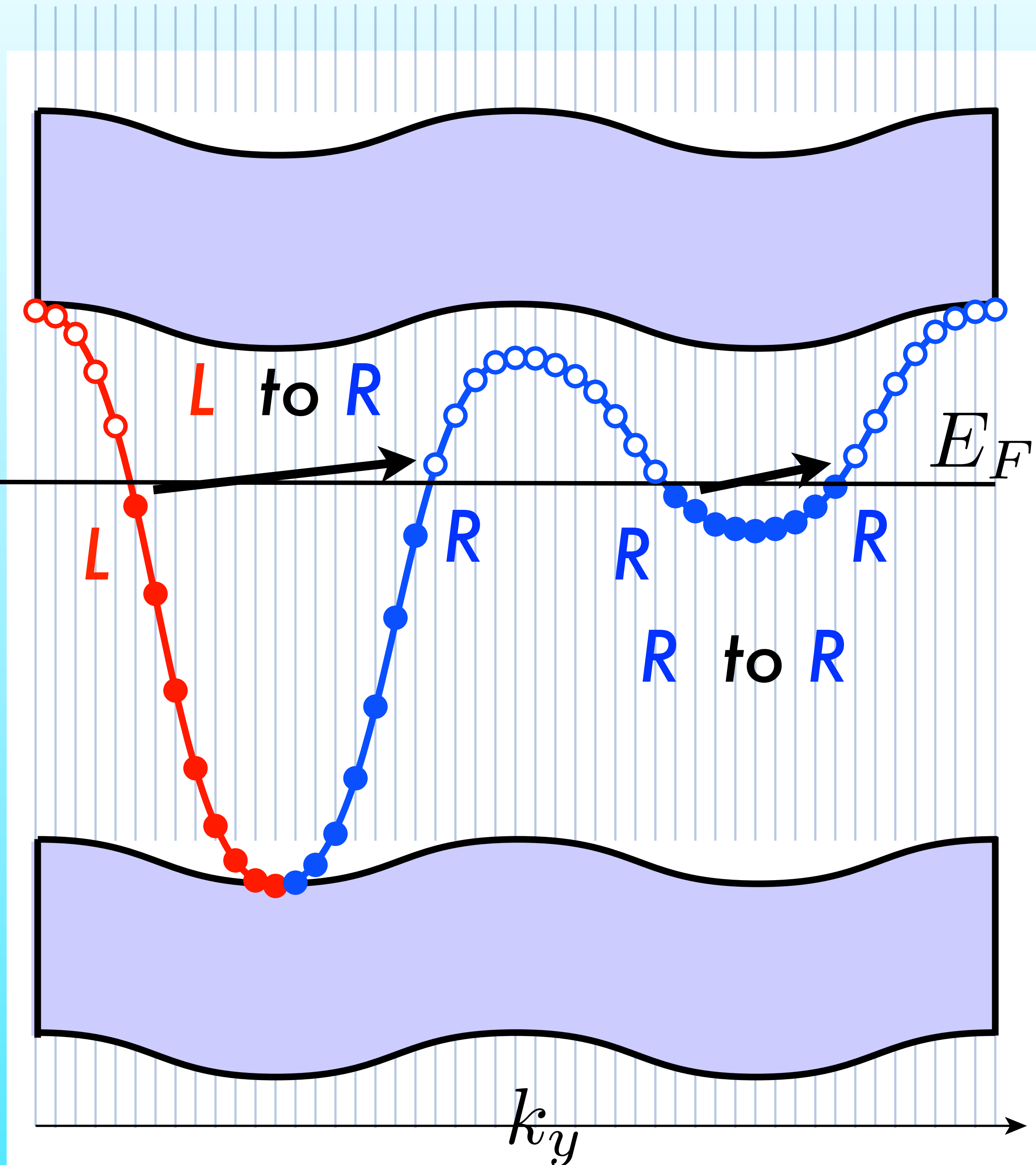

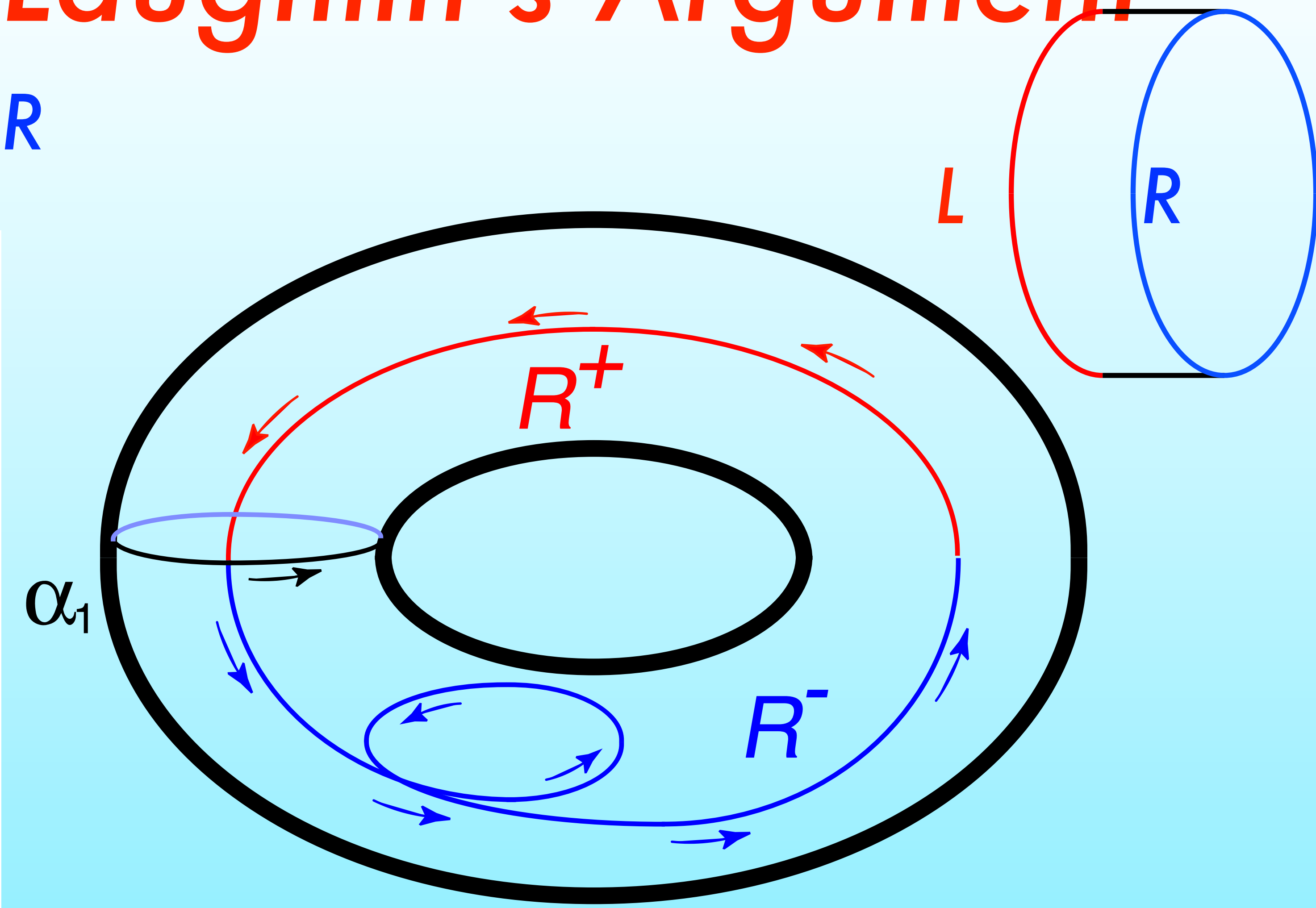

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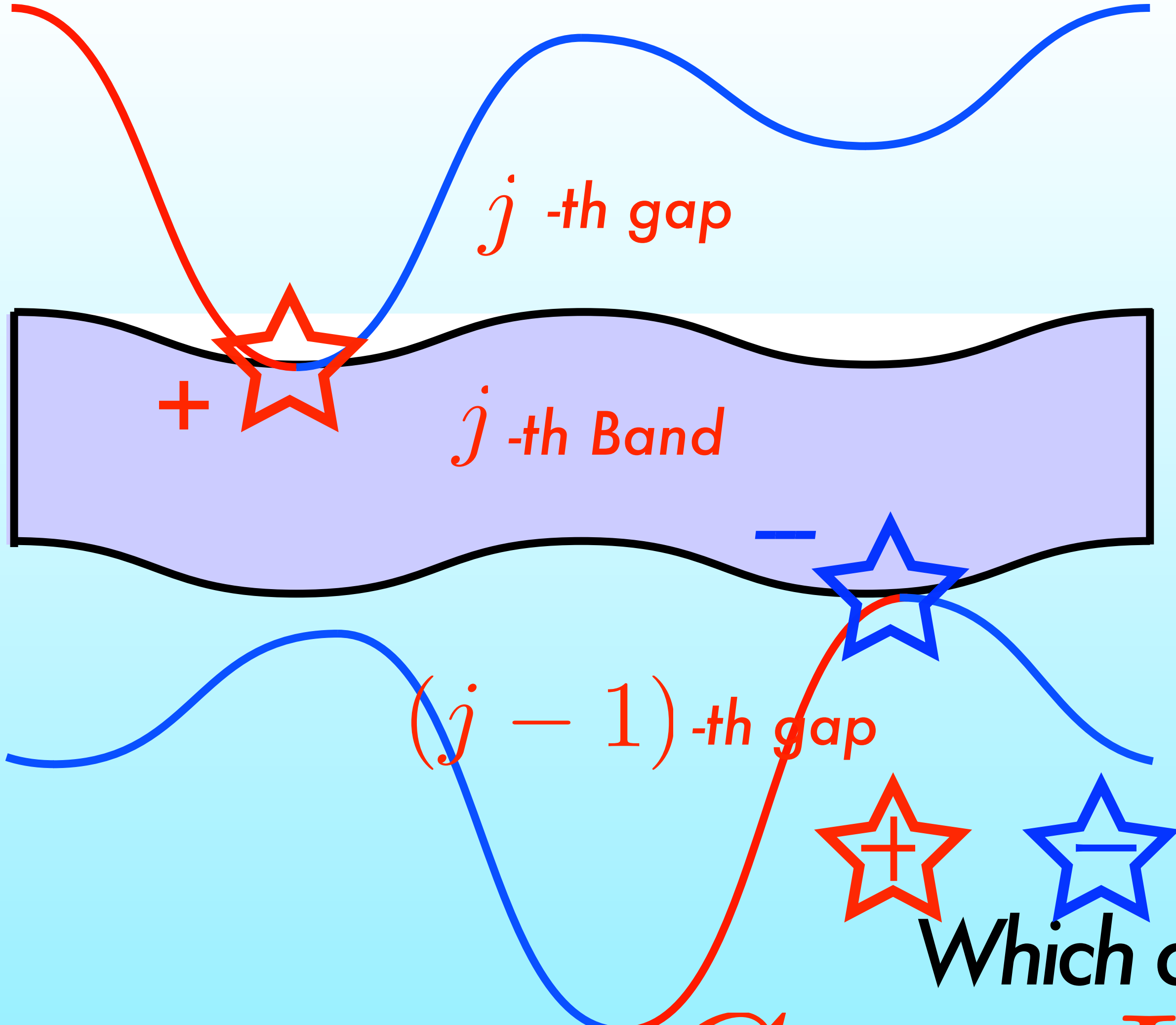
Winding number

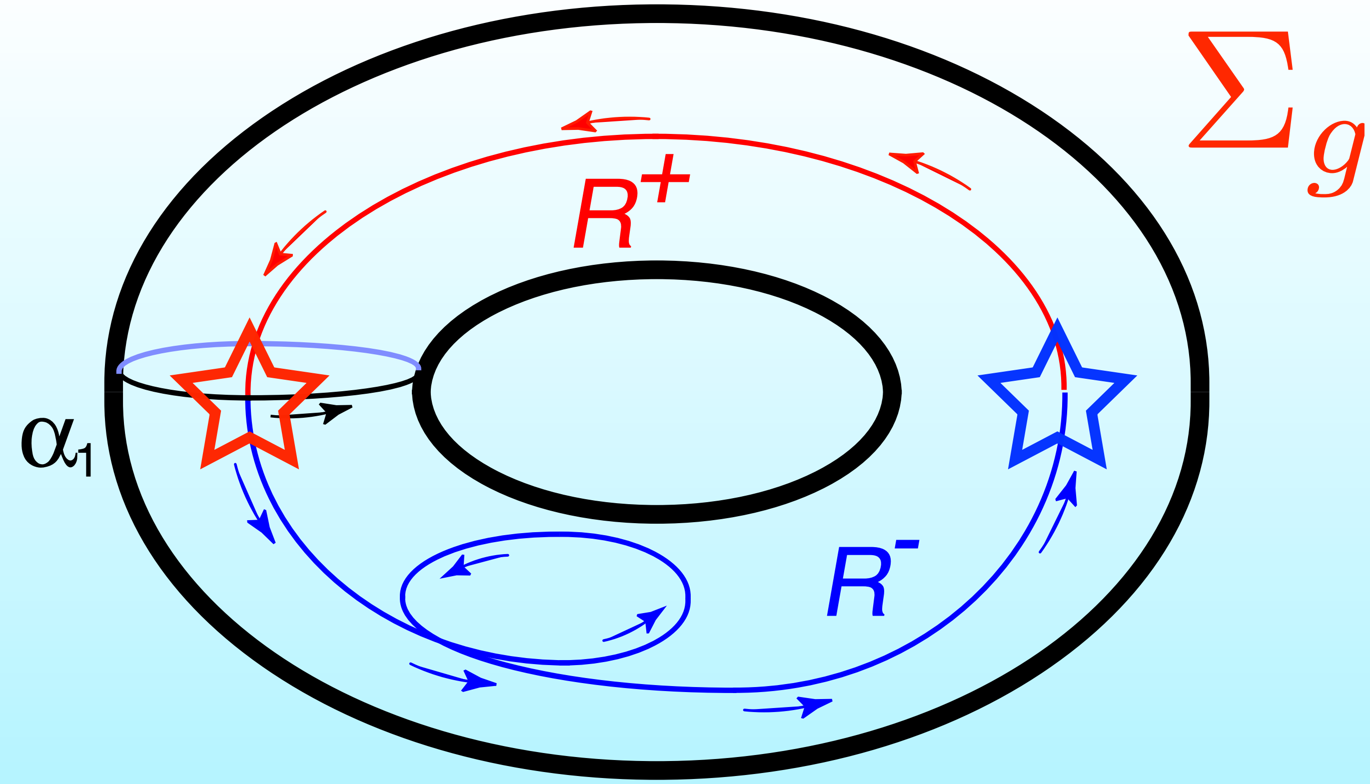
or

Intersection number with canonical loop

Topological number

Y.H., Phys. Rev. B 48, 11851 (1993)





Y.H., Phys. Rev. Lett. 71, 3697 (1993)

Which contribute to the Chern number of the Bulk

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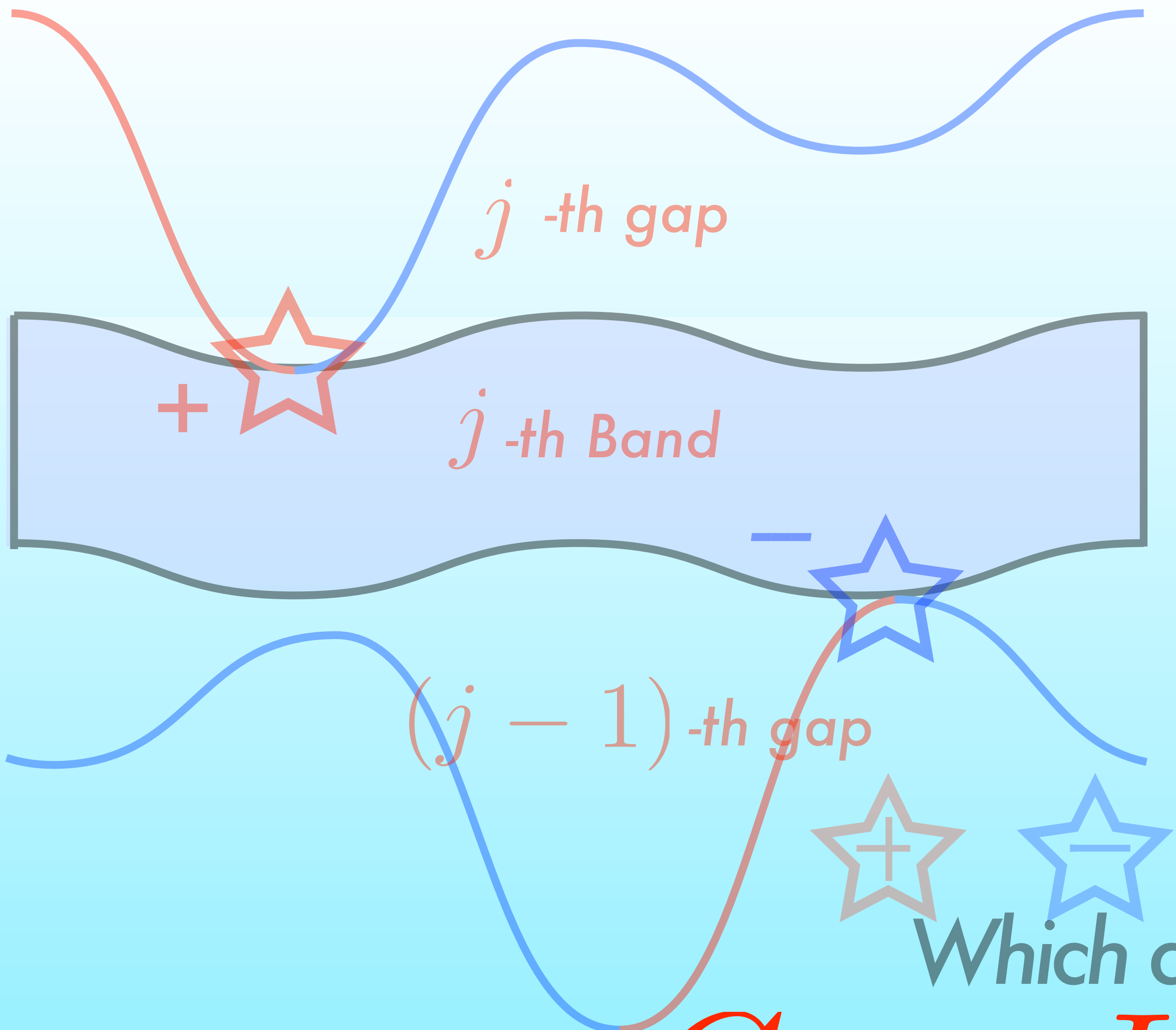
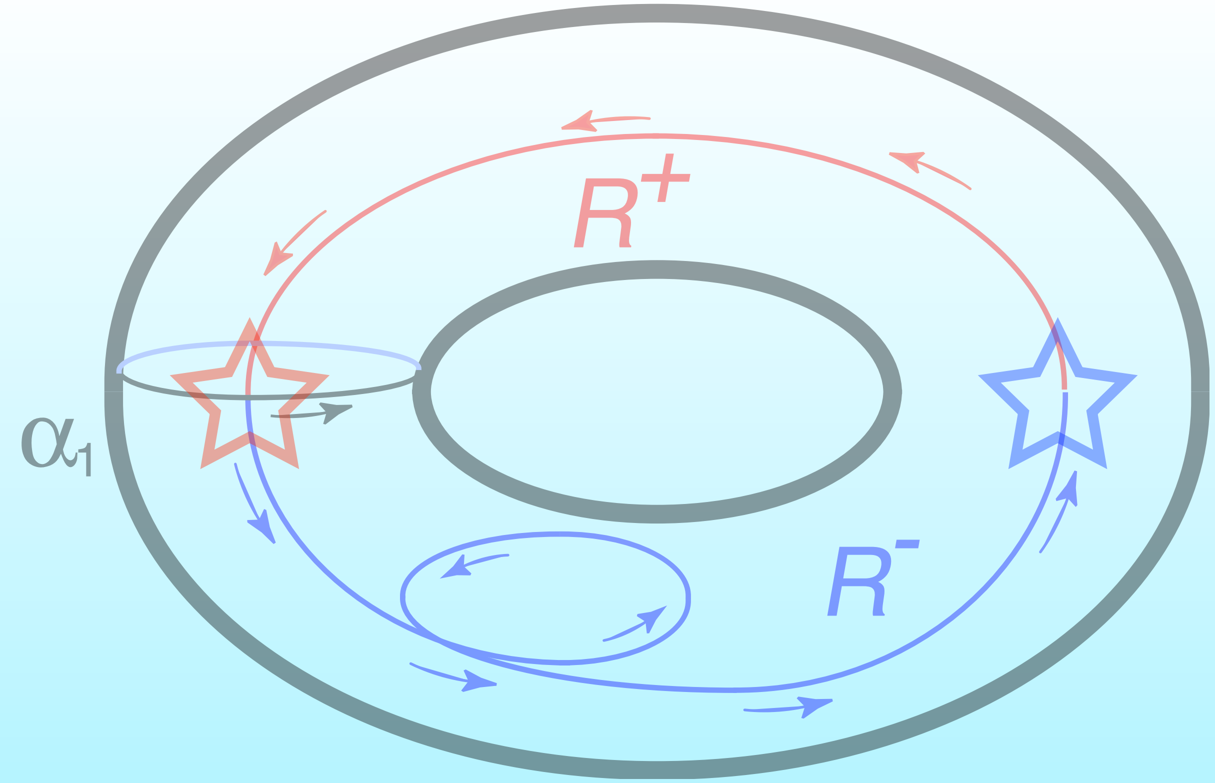
Chern # = winding # Difference between the neighboring gaps

$$\sum_{j=1,\cdots\ell} C_j = I_\ell, \ (\because I_0 = 0)$$

$$\boxed{{\sigma_{xy}}^{\mathrm{bulk}} = {\sigma_{xy}}^{\mathrm{edge}}}$$

Bulk-Edge Correspondence in their topological numbers





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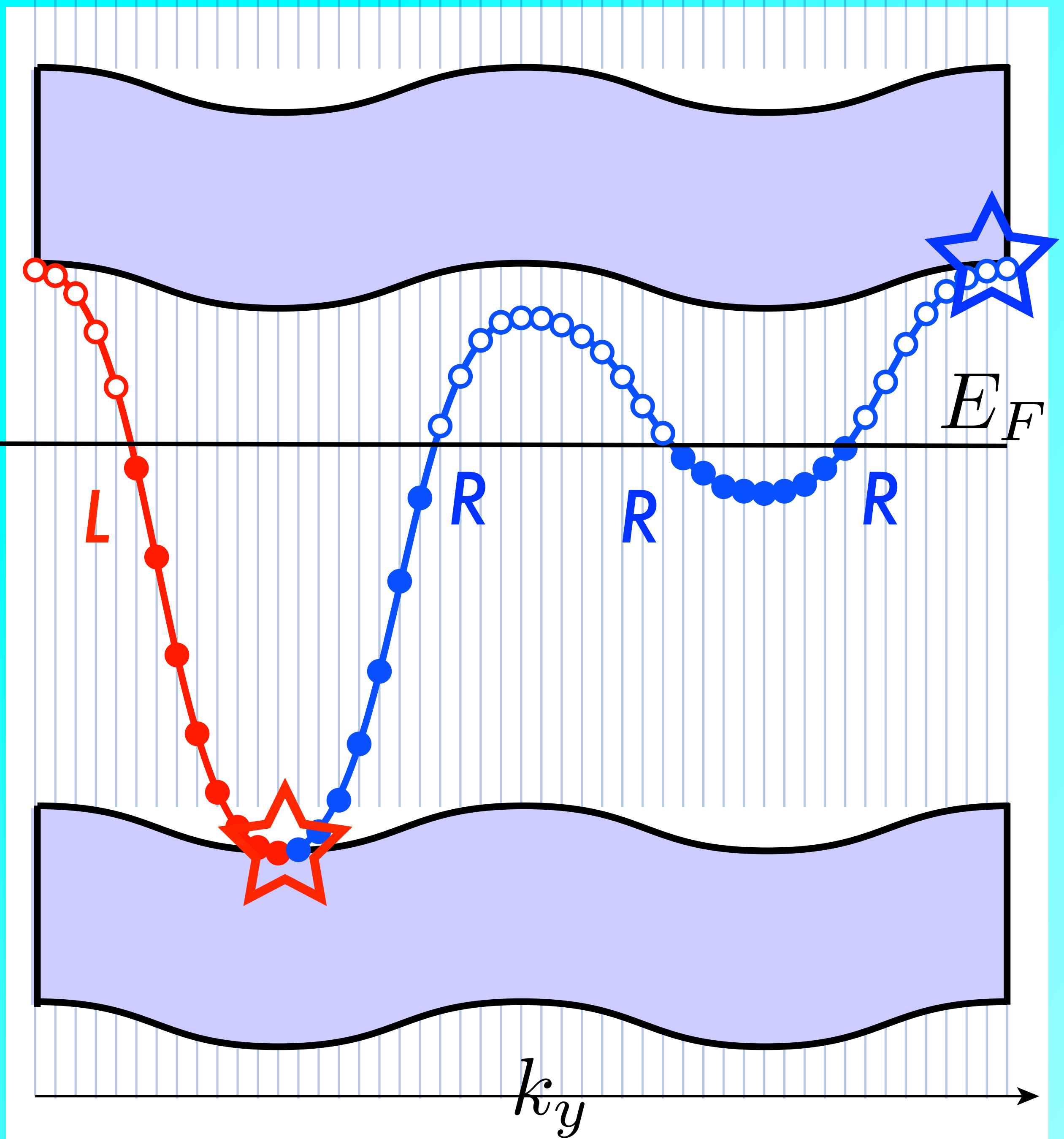
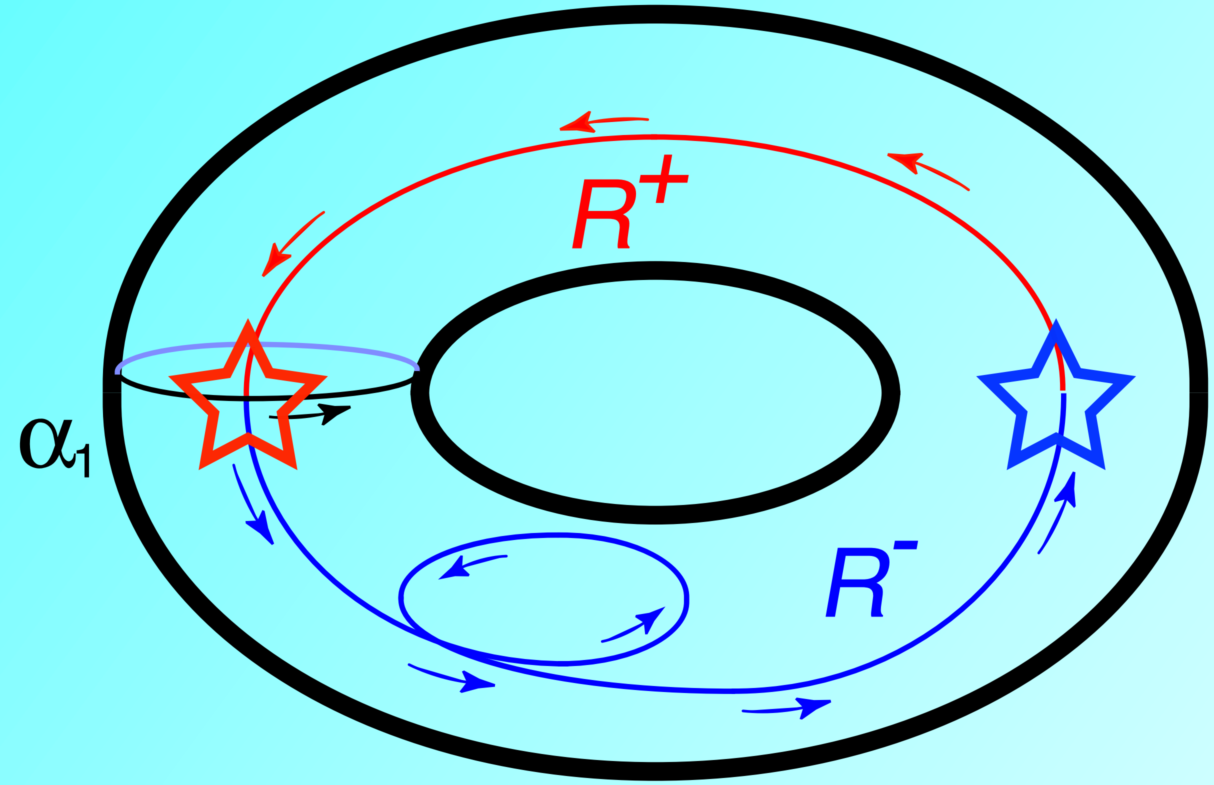




Y.H., Phys. Rev. Lett. 71, 3697 (1993)

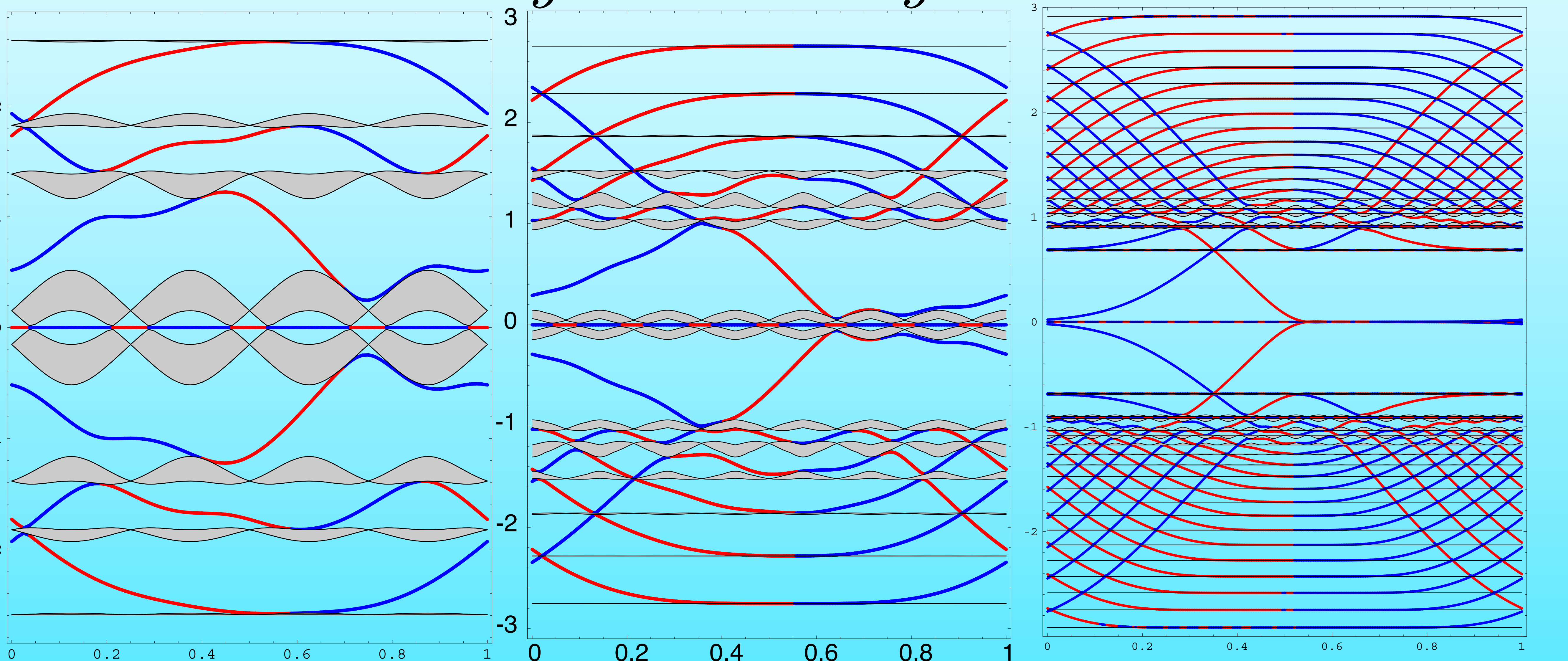
The touching point makes a vortex in the eneryg band

Which contribute to the Chern number of the Bulk

$$C^j_{\mathrm{FS}} = I(\alpha_j, C^j_{\mathrm{edge}})$$

Edge states & Intersection number of Edge State Loops

$$\sigma_{xy}{}^{\text{bulk}} = \sigma_{xy}{}^{\text{edge}}$$



$\phi = 1/4$ $\qquad$ $\phi = 1/7$ $\qquad$ $\phi = 1/21$

Imagine loops on the Riemann surface