This short note is for details of the Laughlin argument (R. B. Laughlin, Phys. Rev. B 23, 5632R (1981)), which is the key arugment for all topological phases.

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I. GAUGE TRANSFORMATION

A. Classical mechanics

Charged particles in EM field

$$L = \frac{1}{2}m\dot{\boldsymbol{r}}^2 - e\phi + \dot{\boldsymbol{r}}\cdot\boldsymbol{A}$$
$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{r}}} = m\cdot\boldsymbol{r} + e\boldsymbol{A}$$
$$H = \boldsymbol{p}\cdot\dot{\boldsymbol{r}} - L = \frac{1}{2m}(\boldsymbol{p}-e\boldsymbol{A})^2 + e\phi$$
$$e\dot{\boldsymbol{r}} = e\boldsymbol{v} = \frac{e}{m}(\boldsymbol{p}-e\boldsymbol{A}) = -\frac{\partial H}{\partial \boldsymbol{A}}$$

Gauge transformation

$$\boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\nabla}\chi, \quad \phi' = \phi - \frac{\partial\chi}{\partial t}$$
$$\boldsymbol{B}' = \mathrm{rot}\,\boldsymbol{A}' = \boldsymbol{B}$$
$$\boldsymbol{E}' = -\frac{\partial \boldsymbol{A}'}{\partial t} - \boldsymbol{\nabla}\phi' = \boldsymbol{E}$$
$$L' = L + e(\dot{\boldsymbol{r}}\cdot\boldsymbol{\nabla}\chi + \frac{\partial\chi}{\partial t}) = L + \frac{d}{dt}\chi(\boldsymbol{r}(t),t)$$
$$\boldsymbol{p}' = \frac{\partial L'}{\partial\dot{\boldsymbol{r}}} = \boldsymbol{p} + e\boldsymbol{\nabla}\chi$$
$$\boldsymbol{p}' - e\boldsymbol{A}' = \boldsymbol{p} - e\boldsymbol{A}$$
$$\dot{\boldsymbol{r}}' = \dot{\boldsymbol{r}}$$
$$H' = H - e\frac{\partial\chi}{\partial t}$$

B. Quantum mechanics

The gauge covariance of the Schrodinger eq. requires

$$H = \frac{1}{2m}(-\mathrm{i}\hbar\boldsymbol{\nabla} - e\boldsymbol{A})^2 + e\phi$$
$$H' = \frac{1}{2m}(-\mathrm{i}\hbar\boldsymbol{\nabla} - e\boldsymbol{A}')^2 + e\phi'$$

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = H\psi$$
$$\mathrm{i}\hbar\frac{\partial\psi'}{\partial t} = H'\psi'$$

$$(\mathrm{i}\hbar\frac{\partial}{\partial t} - e\phi)\psi = \frac{1}{2m}(-\mathrm{i}\hbar\boldsymbol{\nabla} - e\boldsymbol{\nabla}\boldsymbol{A})^2\psi$$
$$(\mathrm{i}\hbar\frac{\partial}{\partial t} - e\phi')\psi' = \frac{1}{2m}(-\mathrm{i}\hbar\boldsymbol{\nabla} - e\boldsymbol{\nabla}\boldsymbol{A}')^2\psi'$$

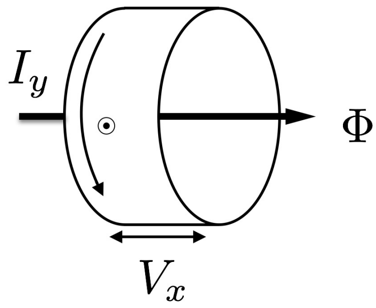

FIG. 1. Laughlin's geometry

Writing the wavefunction as

$$\psi' = \psi\exp(\mathrm{i}\frac{e\chi}{\hbar}) = \psi\exp(\mathrm{i}2\pi\frac{e\chi}{h}) = \psi\exp(\mathrm{i}2\pi\frac{\chi}{\Phi_0})$$

we have

$$(\mathrm{i}\hbar\frac{\partial}{\partial t} - e\phi')\psi' = \exp(\frac{e\chi}{\mathrm{i}\hbar})\Big[\mathrm{i}\hbar\frac{\partial}{\partial t} - e(\phi' + \frac{\partial\chi}{\partial t})\Big]\psi$$
$$= \exp(\frac{e\chi}{\mathrm{i}\hbar})(\mathrm{i}\hbar\frac{\partial}{\partial t} - e\phi)\psi$$
$$(-\mathrm{i}\hbar\boldsymbol{\nabla} - e\boldsymbol{A}')\psi' = \exp(\frac{e\chi}{\mathrm{i}\hbar})\Big[-\mathrm{i}\hbar\boldsymbol{\nabla} - e(\boldsymbol{A}' - \boldsymbol{\nabla}\chi)\Big]\psi$$
$$= \exp(\frac{e\chi}{\mathrm{i}\hbar})(-\mathrm{i}\hbar\boldsymbol{\nabla} - e\boldsymbol{A})\psi$$

It implies the consistency of the gauge transformation.

II. LAUGHLIN ARGUMENT

Let us discuss the situation in Fig.fig:laughlin

A. Byers-Yang formula

Noting that $e<0$, one has for an N partcile system on the cylindrical geomety, electron density $n = N/L_xL_y$ and

$$I_y = -e\langle v_y\rangle nL_x$$

since $-e\langle v_y\rangle$ is a current density and L_x is a section of the cylinder. Here the average velocity of the electrons are estimated as

$$(-e)\langle v_y\rangle = \frac{1}{N}\sum_i^N \frac{\partial H}{\partial A_y} = \frac{1}{N}\frac{\delta E}{\delta A_y}$$

where E is a total energy of the N-electron system. Then one arrive at the formula by Byers-Yang as

$$\begin{aligned} I_y &= \frac{1}{N}\frac{\delta E}{\delta A_y}\frac{N}{L_x L_y}L_x \\ &= \frac{1}{L_y}\frac{\delta E}{\delta A_y} \\ &= \frac{\delta E}{\delta \Phi} \quad : \text{Byers-Yang, PRL 7, 46 (1961)} \end{aligned}$$

where we assume

$$\begin{aligned} A_y &= \text{const.} \\ \Phi &\equiv L_y A_y \end{aligned}$$

See. Fig.1 and the discussion in the next section.

B. AB flux

Introduction of Φ is described by the vector potential $\boldsymbol{A}_\Phi$

$$\oint_{\partial S} d\boldsymbol{r}\cdot\boldsymbol{A}_\Phi = \int_S d\boldsymbol{S}\cdot \mathrm{rot}\,\boldsymbol{A}_\Phi = \Phi$$

where ∂S is a boundary of the cylinder of the length L_y.

Then one can choose $\boldsymbol{A}' = 0$ by taking

$$\boldsymbol{\nabla}\chi = -\boldsymbol{A}_\Phi$$

Note that this is only possible when

$$\mathrm{rot}\,\boldsymbol{A}_\Phi = 0.$$

One form of such a solution is

$$\chi = -\frac{y}{L_y}\Phi,$$

$$\boldsymbol{A}_y = -\boldsymbol{\nabla}\chi = \frac{\hat{y}}{L_y}\Phi = \text{const.}$$

$$\oint_{\partial S} d\boldsymbol{r}\cdot\boldsymbol{A}_\Phi = \oint_0^{L_y} d\hat{y}\cdot\boldsymbol{A}_y = \Phi.$$

Then assuming the periodic boundary condition for ψ,

$$\psi(x, y+L) = \psi(x, y)$$

the gauge transformed wave funstion ψ satisfies

$$\psi'(x, y+L) = e^{\mathrm{i}2\pi\frac{\Phi}{\Phi_0}}\psi'(x, y)$$

although $\boldsymbol{A}_\Phi$ does not apper in the Schrodinger equation for ψ'

$$\begin{aligned} \mathrm{i}\hbar\frac{\partial}{\partial t}\psi' &= \big[\frac{1}{2m}(\boldsymbol{p}-e\boldsymbol{A})^2 + e\phi\big]\psi' \\ \mathrm{i}\hbar\frac{\partial}{\partial t}\psi &= \big[\frac{1}{2m}(\boldsymbol{p}-e(\boldsymbol{A}+\boldsymbol{A}_\Phi)^2 + e\phi\big]\psi' \end{aligned}$$

Generically the flux Φ does modifies the system thorough the boundary condition. However if

$$\Phi = n\Phi_0, \quad n\in\mathbb{Z}$$

the flux Φ does not affect the system.

C. Laughlin argument

When the system is sufficiently large, effects of Φ for the local hamiltonian is $\mathcal{O}(L_y^{-1})$. Then let us consider an adiabatic increase of the Φ. By replacing $\delta\Phi$ to a finite difference $\Delta\Phi$, one has

$$I_y = \frac{\Delta E}{\Delta\Phi}$$

When $\Delta\Phi = \Phi_0$, the system goes back to the original state. Then only a possible modification of the system in the adiabatic process is that $n\in\mathbb{Z}$ electrons are passing through the system. Now we have an estimate

$$\Delta E = neV_x$$

and

$$I_y = \frac{neV_x}{h/e} \equiv \sigma_{yx}V_x$$

it implies

$$\sigma_{yx} = \frac{e^2}{h}n$$

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