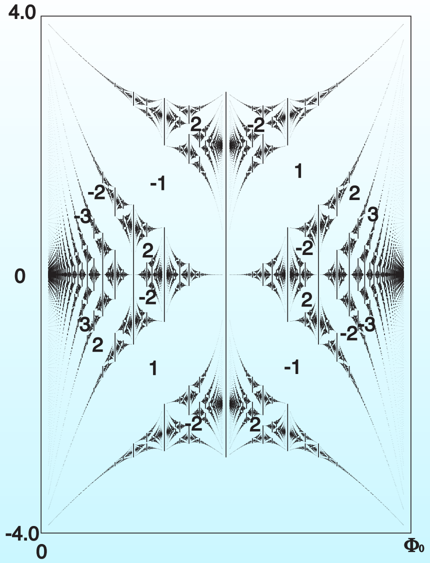
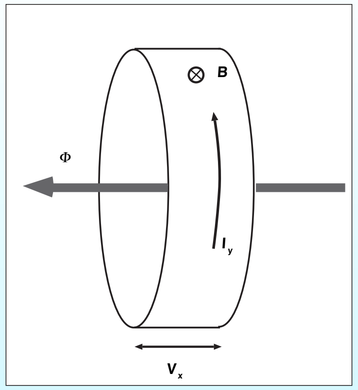
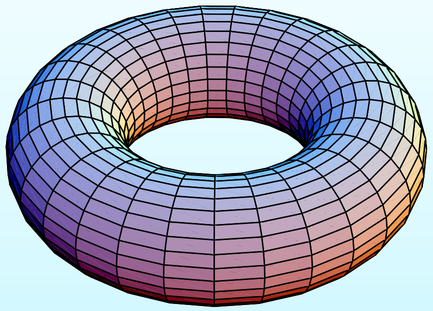
APS March Meeting (2005)







Generic Chern Numbers for a Degenerate Multiplet

For A Characterization of Topological Orders

Y. Hatsugai
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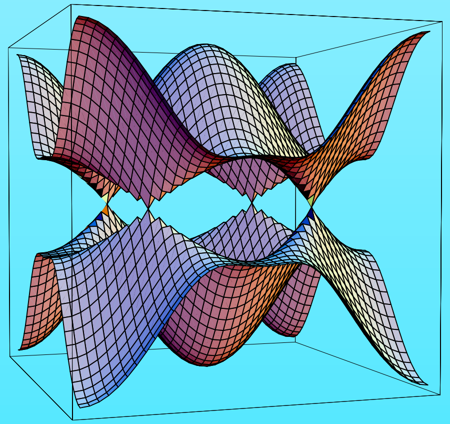

Ref. Y. Hatsugai, cond-mat/0405551, J. Phys. Soc. Jpn.73, 2604 (2004)
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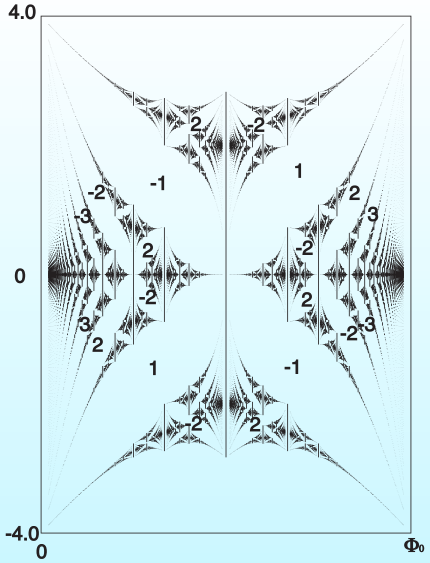

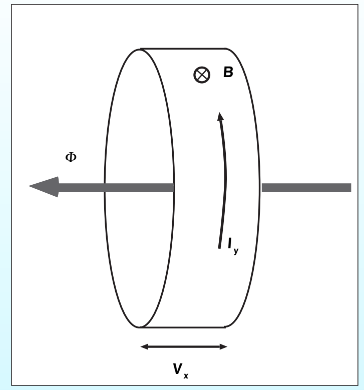

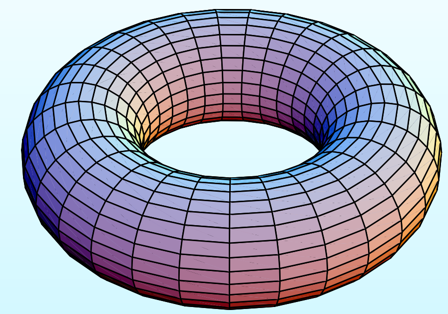

日本物理学会 2005年３月

チャーン数によるトポロジカル秩序相の特徴付け

Generic Chern Numbers for a Degenerate Multiplet
For A Characterization of Topological Orders

初貝安弘

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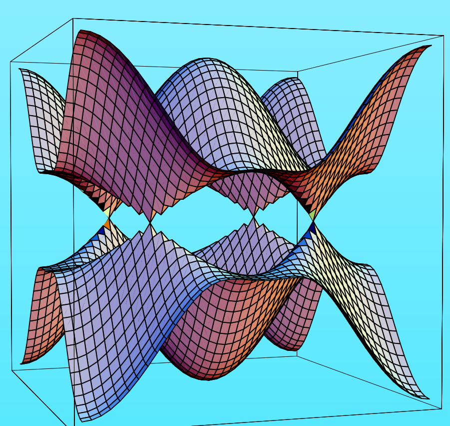

Ref. Y. Hatsugai, cond-mat/0405551, J. Phys. Soc. Jpn.73, 2604 (2004)
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Plan of today's Talk

- Introduction and Motivation
  - How to characterize Quantum Liquids ?
  - Symmetry Breaking (SB) and Topological Order
  - SB and Degeneracy
- Spectral Flow and Berry's Connection
  - Low energy Cluster with Inevitable Degeneracy
- Chern Numbers as Topological Order Parameters
  - Intrinsically Integer
  - Topological Quantities
- Generic Formulation and Specific to Bulk Topological Order
  - Translational Symmetry and Boundary Gauge Twists
- Applications to Quantum Spins (Examples)
  - XXZ models in 2 dimensions
  - Chiral Spin States

Quantum Liquids ?

- Examples (with Energy Gap)
  - Laughlin States (fractional Quantum Hall effects)
  - Integer Quantum Hall States
  - Integer Quantum Spin Chains (Haldane phase)
  - Spin Gap Phases of the t-J model in high-Tc
  - Kondo Lattice at Half filling
  - Dimer Models (Rokhsar-Kivelson,...)
  - String Net Condensations (X.G.Wen)
- Low Dimensional Quantum Systems

Without Any Symmetry Breaking

Strong Quantum Fluctuations

→ Destroy Conventional (local) Orders

How to Characterize Quantum Liquids ?

- Quantum Fluctuations are Not RANDOM!
  - Marshall Sign Rule for Quantum Spins
  - Fractional Statistics of Quasiparticles in FQHE
  - Off Diagonal (quasi) Long Range Order in FQHE
  - String Order parameters in Haldane Spins
- Hidden Rules in the Quantum State

  → Topological Order (X.G.Wen)
- No Symmetry Breaking ! (SB can be secondary)
- No Local Order Parameters to characterize
- Proposal: Use Topological Quantities

  (for some classes)

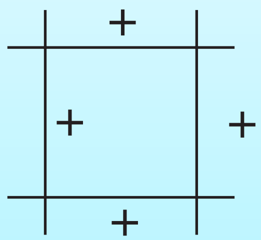
  c.f. quantized Hall conductance for the IQHE

Ex. Characterization of Superconductors: Is Symmetry of the Order Parameter Enough?

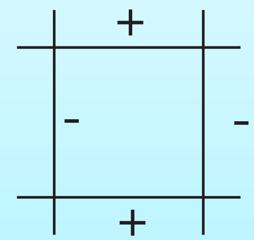
- 2D Examples (Singlet)

No!!

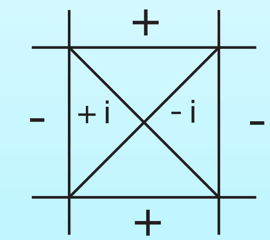
s-wave

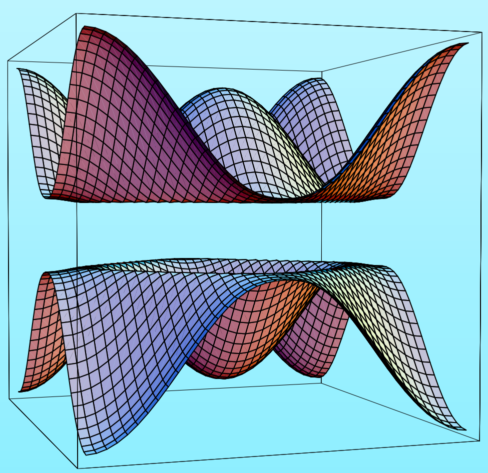


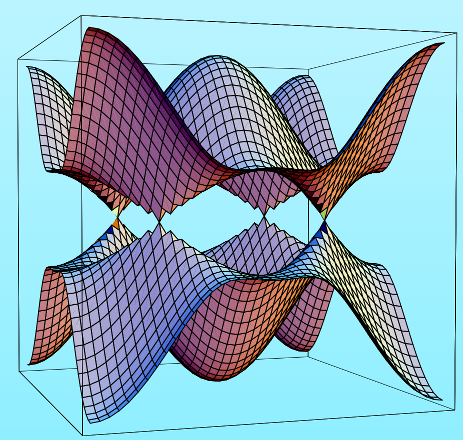
d-wave

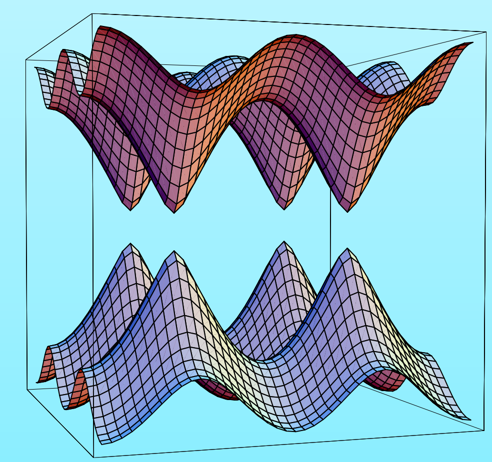


d+id-wave

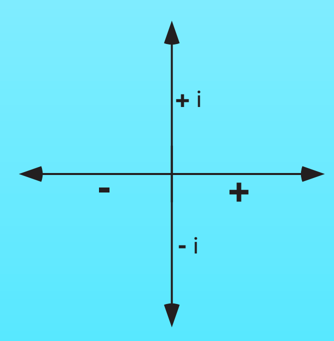


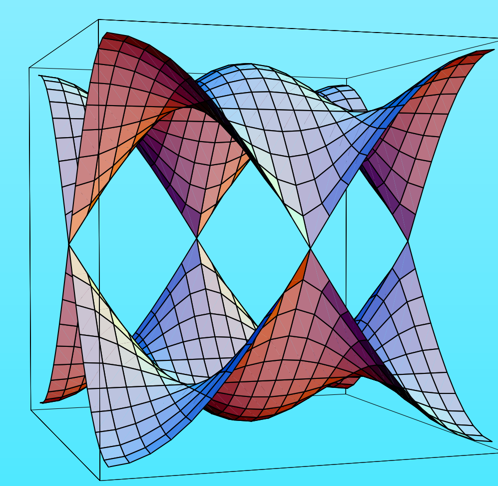






- 2D Examples (Triplet) : Chiral p wave





- Use Spin Hall Conductance to Characterize!

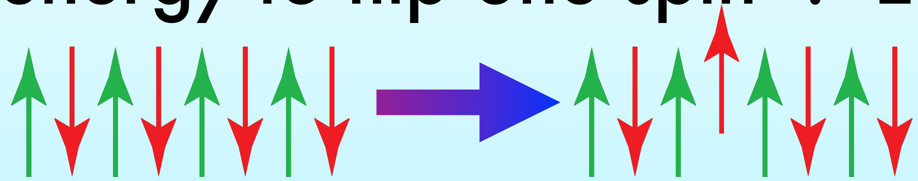
(Intrinsic Topological Meaning)

Topological Quantities to replace (supplement)

- Use Geometrical Phases
  - Need Physical Quantities such as Order Parameters
  - Topological Numbers!
- The Chern Numbers for the Ground State
  - Intrinsically Integer
  - Gauge Independent Quantities with Gauge dependent Definitions
  - Allow Degeneracies (Essential!)
  - Require Generic Energy Gap
  - Topological Insulators

Symmetry Breaking and Degeneracy

- Real Symmetry Breaking (ex. Ising Order)
  - Need finite energy to flip one spin : Energy Gap! $E_g > 0$



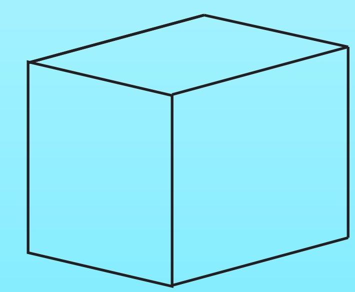
  - Almost Degenerate States in a finite system

Infinite System

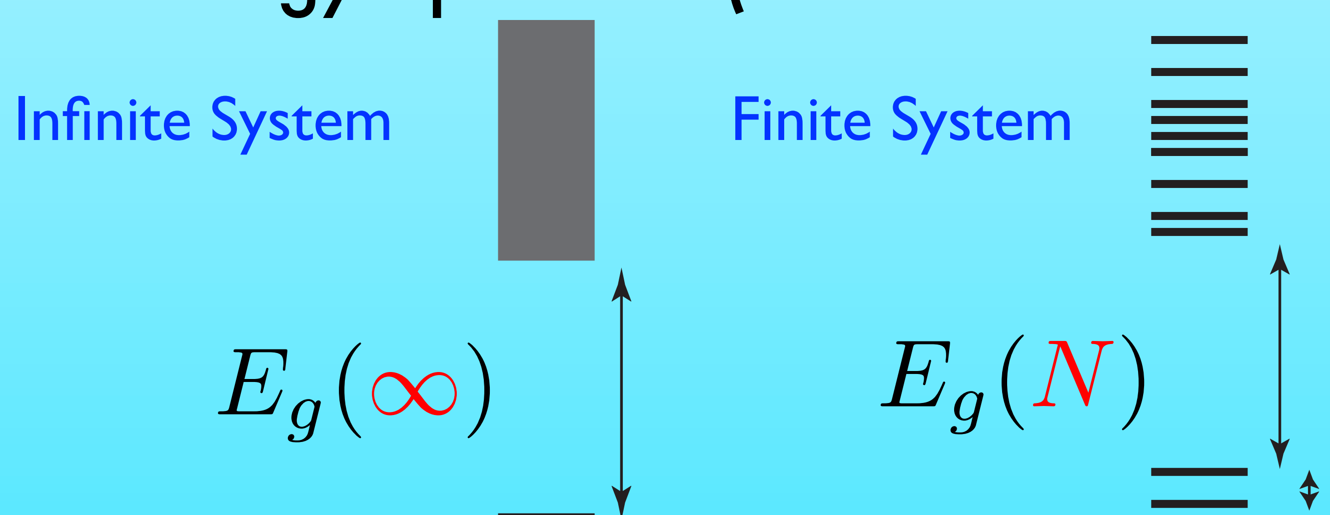
Finite System

Bonding and Anti-bonding states

  - Energy Spectrum (Periodic Boundary Condition)



$N = L^d$



$E_1 \propto e^{-N/\xi^d}$ $c.f.$ $\propto e^{-L/\xi}$ for T.O.

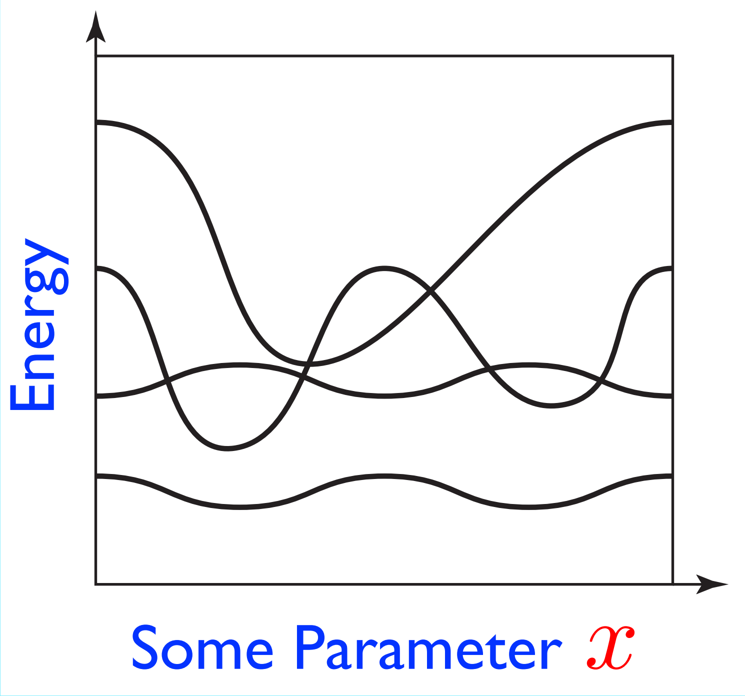
- Similar Degeneracy for Topological Ordered States! (Wen)

  q^g fold degeneracy for the FQH $1/q$ states on a Torus $(g = 1)$

Spectral Flow and Berry's Connection

- Spectral Flow
  - Hamiltonian depends on Some parameters $H(x)$
  - Eigenstate Energies as a Function of Some Parameters $E_n(x)$

$$H(x)|n(x)\rangle = E_n(x)|n(x)\rangle$$



- Berry's Connection

$$\boxed{\begin{aligned} \mathcal{A} &= \langle n|d|n\rangle \\ \vec{A} &= \langle n|\vec{\nabla}|n\rangle \end{aligned}}$$

- Phase Ambiguity → Gauge Freedom

$$\boxed{\begin{aligned} |n(x)\rangle' &= e^{i\theta(x)}|n(x)\rangle \\ \mathcal{A}' &= \mathcal{A} + id\theta \\ \vec{A}' &= \vec{A}' + i\vec{\nabla}\theta \end{aligned}}$$

Chern Numbers:
Total Flux of Berry's Connection

- Fictitious Flux Density

$$\mathcal{B} = d\mathcal{A}$$
$$\vec{B} = \vec{\nabla} \times \vec{A} = \mathrm{rot}\,\vec{A}$$

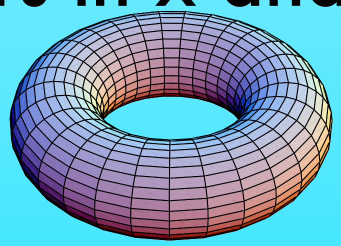
- Total Flux over the Parameter Space is the Chern Number C

$$C = \frac{1}{2\pi i}\int_S \mathcal{B} = \frac{1}{2\pi i}\int_S d\mathcal{A}$$
$$= \frac{1}{2\pi i}\int_S d\vec{S}\cdot\vec{B}$$

- What is the SURFACE S ?

For the QHE, specifying two twisting parameters in x and y directions

$$S = \{(\phi_x, \phi_y) \mid \phi_{x,y} \in [0, 2\pi]\} = T^2$$



Torus
(No Boundary)

- This is Always Integer due to generic Topological Reasons

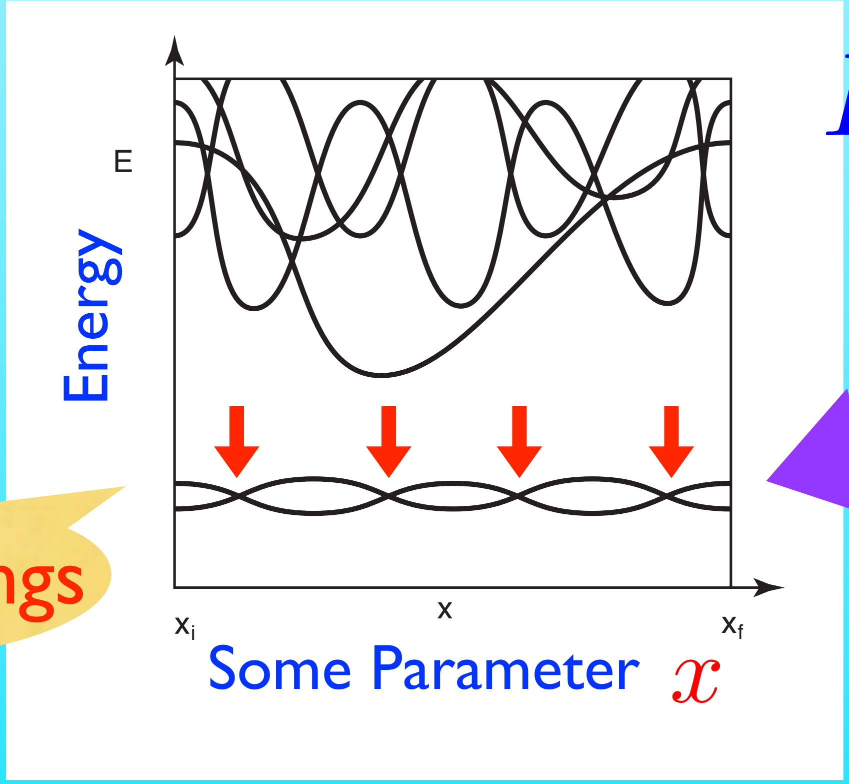
Chern Numbers (Topological Numbers): "Order Parameters" for Quantum Liquids

- Questions ?
  - What is the parameter space?
    - General Theory can not answer!
    - Similar Situations in determining Conventional Order Parameters
- The Parameter Space have to be determined suitably
  - Need Consideration for each order
- Bulk probes for bulk topological Orders
- Local probes for local topological Orders

(Topological) Degeneracy and Low Energy Cluster of Groundstate Multiplet

- (Approximated) Degeneracy of the Ground State
  - Topological Degeneracy
  - Conventional Symmetry Breaking
- Physical Energy Gap is above the ground State Multiplet

Spectral Flow



Level Crossings

$$H(x)|n(x)\rangle = E_n(x)|n(x)\rangle$$

Low Energy Cluster

Ground State Multiplet

Parameter $x \in \mathcal{S} \subset \mathcal{V}$

Ground State Multiplet and Non Abelian Extension

- Can one Define Chern Numbers for Groundstate with Degeneracy?
- Chern Number for a Multiplet of the Low Energy Cluster
- Chern Number for each State is ill-defined due to Level Crossings!
- Berry Connection for the Multiplet (Cluster of M states)

$$H(x)|\psi_j(x)\rangle = E_j(x)|\psi_j(x)\rangle \qquad \mathbf{\Psi}(x) = \Big(|\psi_1\rangle, |\psi_2\rangle, \cdots, |\psi_M\rangle\Big)$$

$$\mathcal{A} = \mathbf{\Psi}^\dagger d\mathbf{\Psi} = \begin{pmatrix} \psi_1^\dagger d\psi_1 & \psi_1^\dagger d\psi_2 & \cdots \\ \psi_2^\dagger d\psi_1 & \psi_2^\dagger d\psi_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Matrix Valued : Non Abelian

Wilczek & Zee

$$C = \frac{1}{2\pi i}\int_S \mathrm{Tr}\, d\mathcal{A}$$

Review : Eguchi-Gilkey-Hanson, Phys. Rep. (1980)
Berry's Conn.:Y.H., J.P.S.J. 73, 2604 (2004)

Examples: Quantum Spins with Twists

related work, Haldane & Arovas, PRB52, 4223 (95)

- How do we choose the parameter Space?
- For picking up Bulk Properties, Translation Invariance is crucial!
- Use 2Dim. Boundary Twists for the parameters! (c.f. Lieb-Mattis)
  - If the interactions are gauge string type, the twists are distributed uniformly preserving translational invariance!
  - It does not change "Bulk Properties" since it is just a surface term.

$$\boxed{H^T(\boldsymbol{\theta})} = \sum_{\boldsymbol{m}} T'^{\boldsymbol{m}} h_\ell^\theta(\boldsymbol{S}'_\theta(\boldsymbol{r}_1), \boldsymbol{S}'_\theta(\boldsymbol{r}_2), \cdots) T'^{\boldsymbol{m}\dagger} \qquad T_\mu^{L_\mu} = 1$$

Periodic B.C.

Translation : $T_\mu \boldsymbol{S}(\boldsymbol{r}) T_\mu^\dagger = \boldsymbol{S}(\boldsymbol{r} + \boldsymbol{a}_\mu)$

Twist : $\boldsymbol{S}'_\theta(\boldsymbol{r}_\eta^{\boldsymbol{m}}) = \boldsymbol{Q}(\gamma) \boldsymbol{S}(\boldsymbol{r}_\eta^{\boldsymbol{m}}) \quad \boldsymbol{Q}(\gamma) = e^{\gamma \boldsymbol{X}} \quad X^{\alpha\beta} = \frac{1}{2} i n^\gamma \mathrm{Tr}\, \sigma^\alpha \sigma^\beta \sigma^\gamma,$

$\gamma = \boldsymbol{m} \cdot \boldsymbol{\theta}$

Unitary (Gauge) Equivalent

$$\boxed{H^P} = H(h_\ell) = \sum_{\boldsymbol{m}} T^{\boldsymbol{m}} h_\ell((\boldsymbol{S}(\boldsymbol{r}_1), \boldsymbol{S}(\boldsymbol{r}_2), \cdots) T^{\boldsymbol{m}\dagger},$$

$$T'^{L_\mu}_\mu = \exp(-\hat{n}_\mu \theta_\mu L_\mu \boldsymbol{X})$$

Twisted B.C.

Some of Local Hamiltonians

- Conditions for string type interactions

$$
\begin{aligned}
h_\ell(\boldsymbol{S}(\boldsymbol{r}_1), \boldsymbol{S}(\boldsymbol{r}_2), \cdots) =& h^G(\boldsymbol{\theta} = \boldsymbol{0}; \{\boldsymbol{r}_i - \boldsymbol{r}_j\}; \boldsymbol{S}(\boldsymbol{r}_1), \boldsymbol{S}(\boldsymbol{r}_2), \cdots) \\
=& h^G(\boldsymbol{\theta}; \{\boldsymbol{r}_i - \boldsymbol{r}_j\}; \boldsymbol{S}'_\theta(\boldsymbol{r}_1), \boldsymbol{S}'_\theta(\boldsymbol{r}_2), \cdots) \\
\equiv& h^\theta_\ell(\boldsymbol{S}'_\theta(\boldsymbol{r}_1), \boldsymbol{S}'_\theta(\boldsymbol{r}_2), \cdots)
\end{aligned}
$$

$$
\boldsymbol{S}(\boldsymbol{r}) = \begin{pmatrix} S_x(\boldsymbol{r}) \\ S_y(\boldsymbol{r}) \\ S_z(\boldsymbol{r}) \end{pmatrix} \qquad \boldsymbol{Q}^z(\gamma) = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \gamma = \boldsymbol{m}\cdot\boldsymbol{\theta}
$$

XXZ type

$$
h_\ell^{\text{pair}} = {}^t\boldsymbol{S}(\boldsymbol{r}_1)\boldsymbol{J}\boldsymbol{S}(\boldsymbol{r}_2) \qquad \boldsymbol{J} = J\text{diag}\,(1,1,\lambda)
$$

$$
h^{\text{pair}} = \tfrac{J}{2}(e^{-i(\theta_1-\theta_2)}S_1{}^+_{\theta_1}S_2{}^-_{\theta_2} + h.c.) + \lambda S_1{}^z_{\theta_1}S_2{}^z_{\theta_2}
$$

Chiral Spin type

$$
h_\ell^{\text{sb}} = J_c\boldsymbol{S}(\boldsymbol{r}_1)\cdot\big(\boldsymbol{S}(\boldsymbol{r}_2)\times\boldsymbol{S}(\boldsymbol{r}_3)\big) = J_c\epsilon_{ijk}\boldsymbol{S}_i(\boldsymbol{r}_1)\boldsymbol{S}_j(\boldsymbol{r}_2)\boldsymbol{S}_k(\boldsymbol{r}_3)
$$

$$
h^{\text{sb}} = \tfrac{J_c}{2}S_1{}^z_{\theta_1}(ie^{-i(\theta_2-\theta_3)}S_2{}^+_{\theta_2}S_3{}^-_{\theta_3} + h.c.) + (\text{cyclic perm.}),
$$

Ex. Apply it for 2Dim. Anti. F. XXZ models

☆ With Ising Anisotropy, Approximately Doubly degenerate in a finite system: Conventional Symmetry Breaking

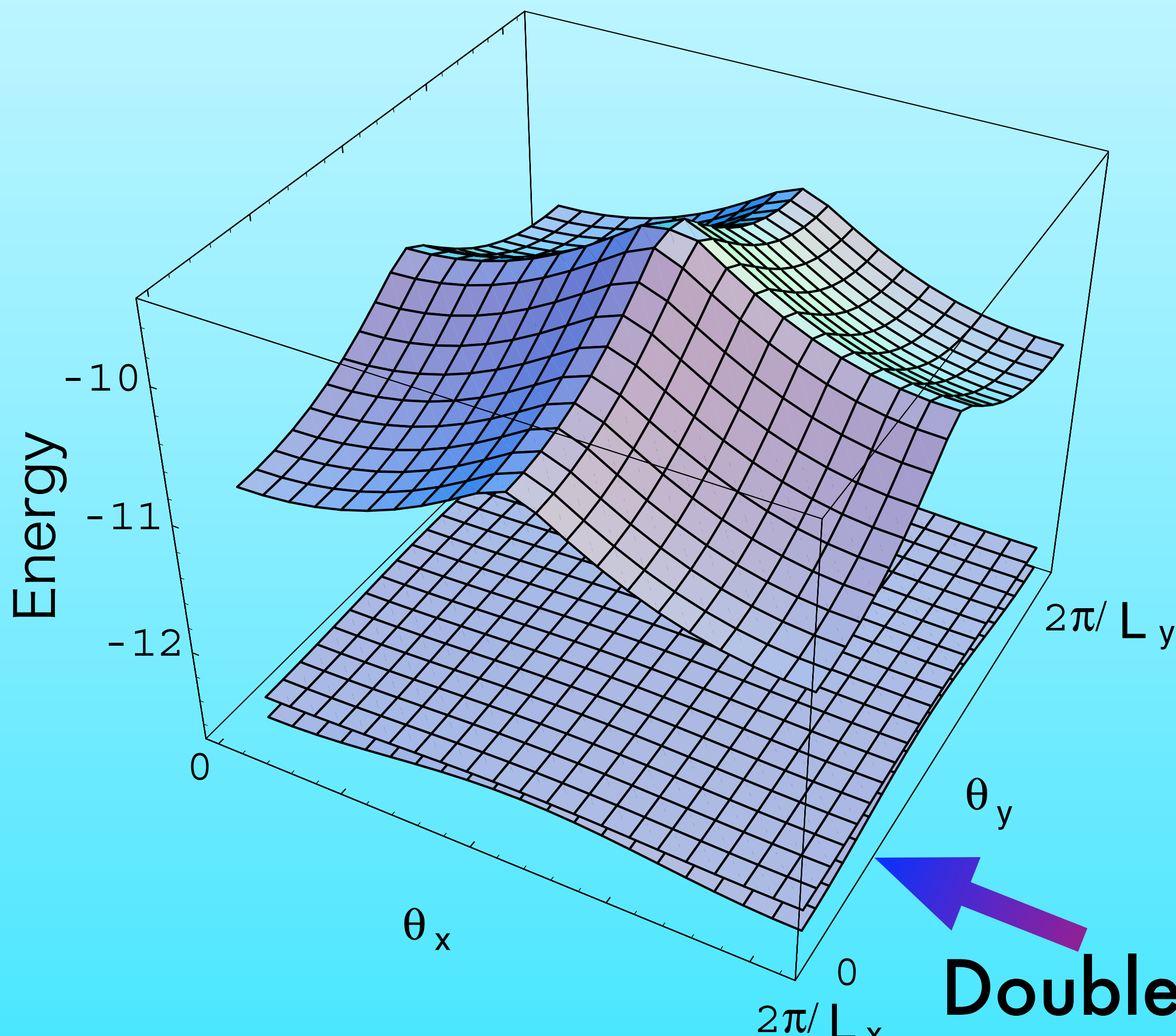
C=0

$$C_S = \frac{1}{2\pi}\oint_{\partial S_0^{\Phi}} d\Omega = N_\Omega(S_0^{\Phi})$$

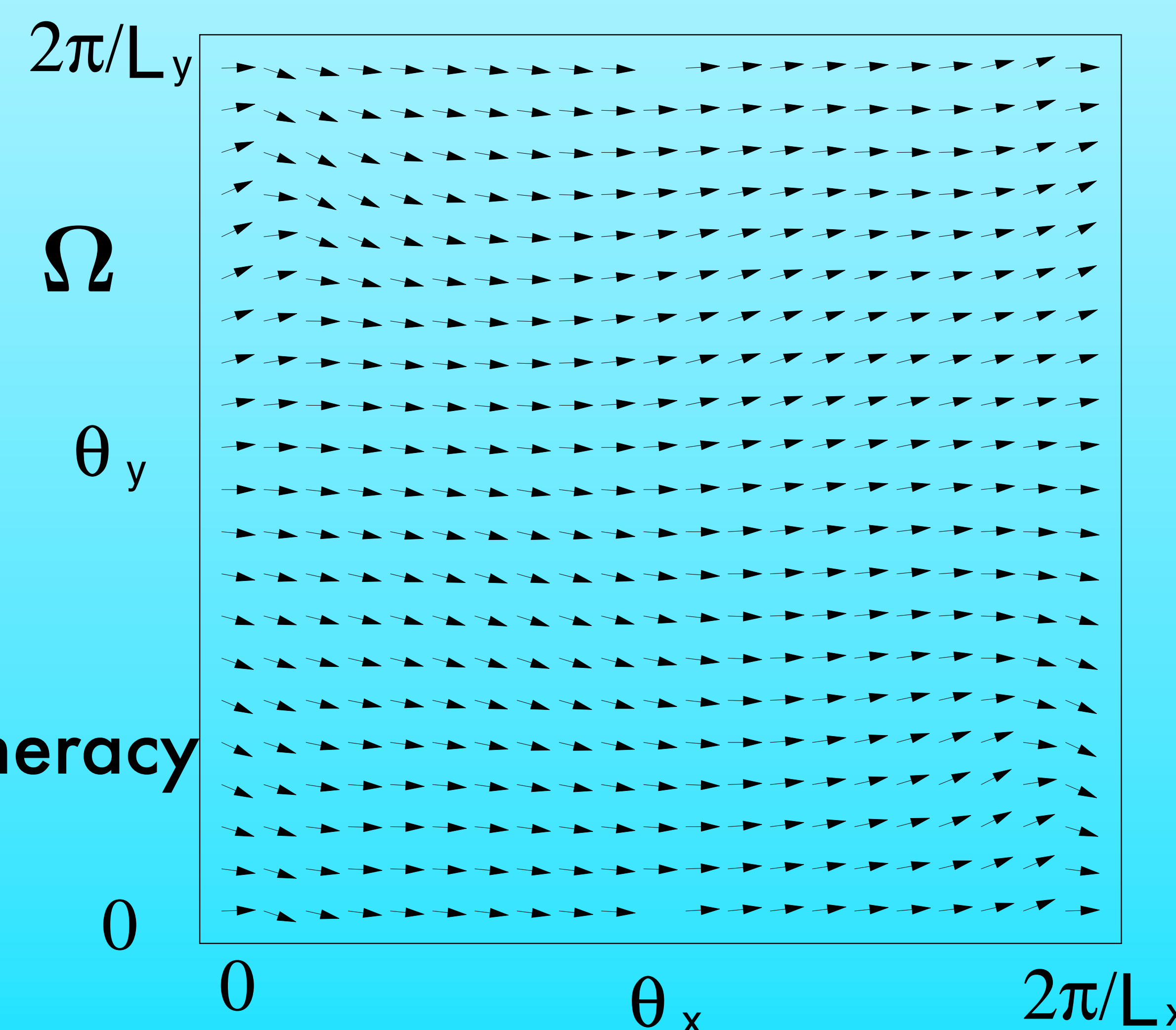
The Lowest 3 Energies with 2Dim. Twists

$$\Omega = \mathrm{Im}\,\mathrm{Tr}\,\log \tilde{\boldsymbol{\Phi}}^\dagger \boldsymbol{P}\boldsymbol{\Phi}$$

$$= \mathrm{Arg}\,\det \tilde{\boldsymbol{\Phi}}^\dagger \boldsymbol{P}\boldsymbol{\Phi}$$



Double Degeneracy



4×4 XXZ model $\quad J_\perp / J_\parallel = 1.3$

With Equal Spin Pairing

Also useful for: Unitary Superconductors

- Eigen states are Doubly degenerate

Parameter x: Momentum k

$$\boldsymbol{H}\psi = E\psi, \qquad \boldsymbol{H} = \begin{pmatrix} \epsilon \boldsymbol{I}_2 & \boldsymbol{\Delta} \\ \boldsymbol{\Delta}^\dagger & -\epsilon \boldsymbol{I}_2 \end{pmatrix}$$:4×4 Matrix

$$\boldsymbol{\Delta} \equiv |\Delta| \boldsymbol{\Delta}_0, \quad |\Delta| \geq 0, \quad \boldsymbol{\Delta}\boldsymbol{\Delta}^\dagger = |\Delta|^2 \boldsymbol{I}_2$$

$$\boldsymbol{\Delta}_0 : 2 \times 2 \text{ unitary matrix,}$$

$$E = -R, R = \sqrt{\epsilon + |\Delta|^2}$$

$$\psi(\boldsymbol{w}) = \begin{pmatrix} -\sin\frac{\theta}{2}\boldsymbol{w} \\ \cos\frac{\theta}{2}\boldsymbol{\Delta}_0^\dagger \boldsymbol{w} \end{pmatrix}$$

$$\boldsymbol{w} = \begin{pmatrix} * \\ * \end{pmatrix}, \boldsymbol{w}^\dagger \boldsymbol{w} = 1$$

Arbitrary 2 dim. vector

$$\boldsymbol{\Delta}\boldsymbol{\Delta}^\dagger = (\boldsymbol{\Delta}^\dagger \boldsymbol{\Delta})^* \text{ \& hermite} : \begin{cases} \propto \boldsymbol{I}_2 & : \text{Unitary} \\ \text{otherwise} & : \text{Non Unitary} \end{cases}$$

$$\boxed{E = \pm\sqrt{\epsilon^2 + |\vec{d}|^2 \pm q}}$$

$$= \begin{cases} |\psi|^2 & : \text{singlet(Unitary)} \\ |\vec{d}|^2 + \vec{\sigma}\cdot\vec{q} & : \text{triplet} \quad \vec{q} = i\vec{d}\times\vec{d}^* \begin{cases} \boxed{\vec{q} = 0 \quad : \text{Unitary}} \\ \vec{q} \neq 0 \quad : \text{Non Unitary} \end{cases} \end{cases}$$

Sum Rule and *Quantum Phase Transition*

- Direct Sum of the Linear Space $W = W_1 \oplus W_2$

$$\mathbf{\Psi} = (\mathbf{\Psi}_1, \mathbf{\Psi}_2) = \left[\begin{array}{c|c} \mathbf{\Psi}_1 & \mathbf{\Psi}_2 \end{array}\right] \qquad \boldsymbol{\mathcal{A}} = \mathbf{\Psi}^\dagger d\mathbf{\Psi} = \left[\begin{array}{c|c} \boldsymbol{\mathcal{A}}_1 & \\ \hline & \boldsymbol{\mathcal{A}}_2 \end{array}\right]$$

$$\mathrm{Tr}\,\boldsymbol{\mathcal{A}} = \mathrm{Tr}\,\boldsymbol{\mathcal{A}}_1 + \mathrm{Tr}\,\boldsymbol{\mathcal{A}}_2$$

$$\boldsymbol{\mathcal{A}}_1 = \mathbf{\Psi}_1^\dagger d\mathbf{\Psi}_1, \quad \boldsymbol{\mathcal{A}}_2 = \mathbf{\Psi}_2^\dagger d\mathbf{\Psi}_2$$

- Sum Rule of the Chern Number (ex. Multi Landau Levels)

$$C_S(W_1 \oplus W_2) = C_S(W_1) + C_S(W_2)$$

- Quantum Phase Transition

$$W_1 \oplus W_2 \leftrightarrows W_1 \oplus W_3$$

$$C_S(W_1) + C_S(W_2) \leftrightarrows C_S(W_1) + C_S(W_3)$$

Summary

- Proposal for characterization of Quantum Liquids ?
  - Symmetry Breaking (SB) and Topological Order
  - SB and Degeneracy
- Chern Numbers as Topological Order Parameters
- Spectral Flow of Low Energy Cluster (Groundstate Multiplet)
- Non-Abelian Berry's Connection of the grd. st. Multiplet
- Generic Formulation for Bulk Topological Orders
- Translational Invariance and Boundary Gauge Twists

Questions

How to Choose a Suitable Parameter Space
for
Each Topological Ordered State ?

Ref. Y. Hatsugai, cond-mat/0405551, J. Phys. Soc. Jpn.73, 2604 (2004)
cond-mat/0412344, to appear in J. Phys. Soc. Jpn.(2005)

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