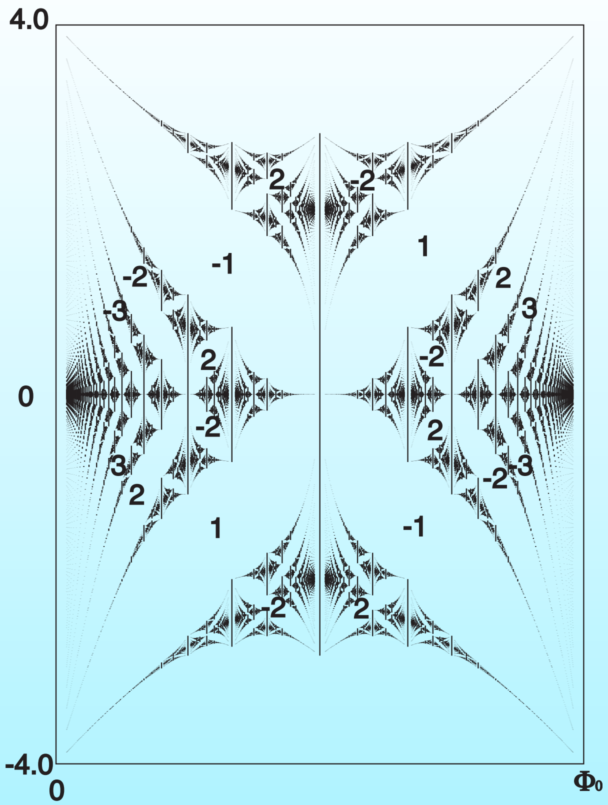
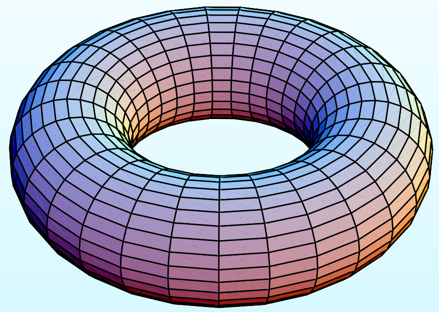
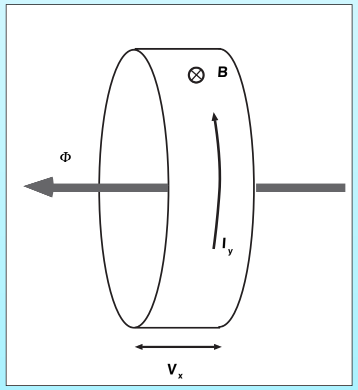


APS March Meeting (2004)



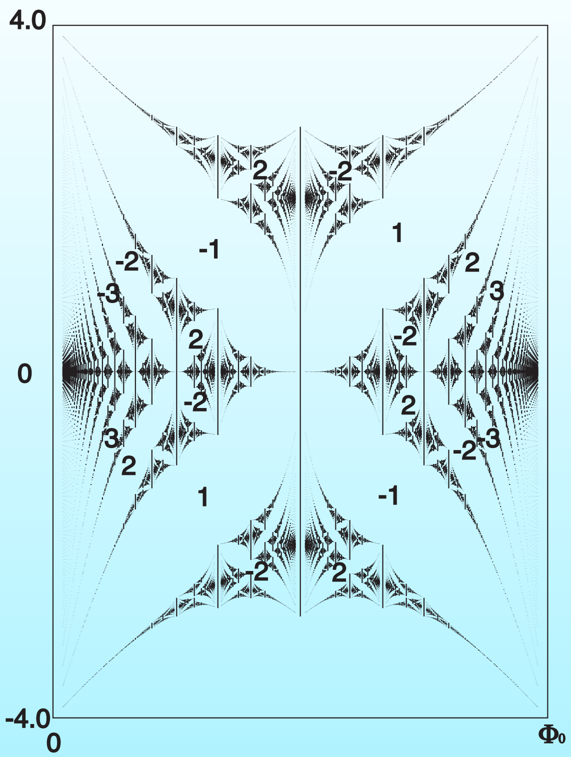
Superconductivity and Chiral Anomalies

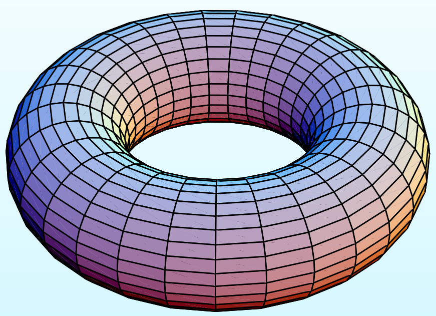


Y. Hatsugai
Department of Applied Physics
University of Tokyo
with S. Ryu and M. Kohmoto

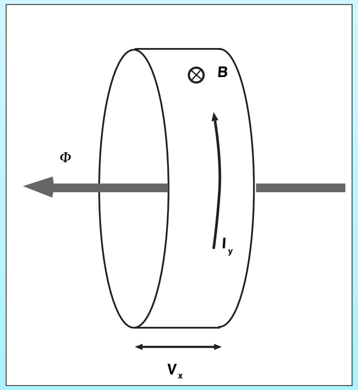
Ref. Y. Hatsugai, S. Ryu and M Kohmoto, cond-mat/038332

JPS Spring Meeting (2004)





Topological Consideration of Generic Superconductivity



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Ref. Y. Hatsugai, S. Ryu and M Kohmoto, cond-mat/038332

Plan of today's Talk

- Topological Order
  - Universal ?   Useful?
  - Previous work: without equal spin pairing (ESP)
- Applications
  - Node Structures of Energy Gap (Topological Origin)
  - Nontrivial Boundary Effects (Edge States)
- Generic BCS Hamiltonian on Lattice (With ESP)
  - Pairing Hamiltonian
  - Symmetries
- Topological Consideration with ESP
  - Non-Abelian Gauge Structures
  - First Chern Numbers (Chiral Anomalies)

Topological Orders

- Quantum Hall Effects (Integer & Fractional)
  - Energy Gap and Quatization of Hall Conductance
  - Bulk v.s. Edge
- Haldane Spin Chain with Integral Spin
  - Energy Gap with Topological Origin
  - Kennedy Triplet with finite Length Spin Chains (Edge States)
- Polyacetylene (SSH model)
- High-Tc (Chiral Spin State, Anyon-superconductivity)
- Chirality Order in Itinerant Magnetism
- Polarization of Insulators, KSV formula
- Anisotropic Superconductivity & Superfluidity
  - Dirac Monopoles and Topological Origin of Zero Bias Peak
  - Topological Origin of Gap Nodes of Superconductor
- Carbon Nano-Structures

Topological Aspects of Supr. Without ESP

YH & SR, PRB65, 212510-1-4(2002)

☆ BCS Hamiltonian and Parametrization

$$\boldsymbol{h}(\boldsymbol{k}) = \begin{pmatrix} \epsilon(\boldsymbol{k}) & \Delta(\boldsymbol{k}) \\ \Delta^*(\boldsymbol{k}) & -\epsilon(\boldsymbol{k}) \end{pmatrix} = \boldsymbol{\sigma}\cdot\boldsymbol{R}(\boldsymbol{k}) : \ \ \boldsymbol{k} \to \boldsymbol{R} = (R_x, R_y, R_z)$$

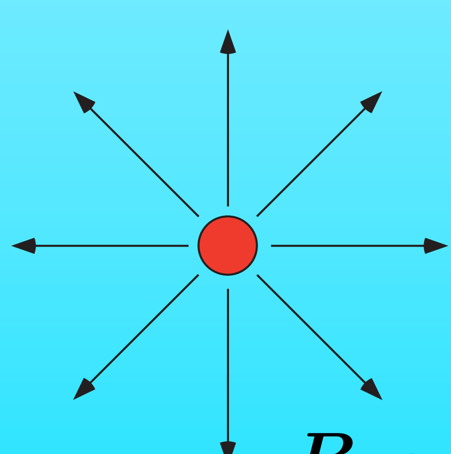
$$\boldsymbol{h}|R\rangle = E_\pm|R\rangle, \quad E_\pm = \pm|\boldsymbol{R}|, \quad |R\rangle = \begin{pmatrix} -\sin\frac{\theta}{2} \\ e^{i\phi}\cos\frac{\theta}{2} \end{pmatrix}$$

☆ Chern Number and Dirac Monopole

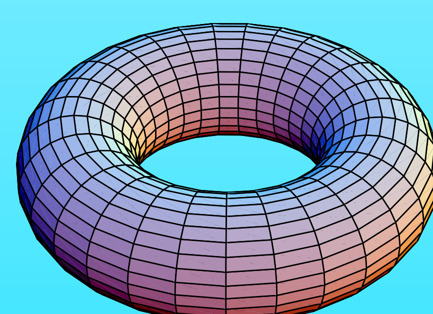
$$\boxed{C = \frac{1}{2\pi i}\int_{BZ} d\boldsymbol{S}_k\cdot\boldsymbol{B}_k} = \frac{1}{2\pi i}\int_{\mathrm{BZ}} dk_x k_y \Big(\langle\partial_x k|\partial_y\rangle - \langle\partial_y k|\partial_x\rangle\Big)$$

$$= \frac{1}{2\pi i}\int_{R(BZ)} d\boldsymbol{S}_R\cdot\boldsymbol{B}_R, \quad \boldsymbol{B}_R = \mathrm{rot}_R\ , \boldsymbol{A}_R = -\frac{i}{2}\frac{\boldsymbol{R}}{R^3}, \quad \boldsymbol{A}_R = \langle R|\boldsymbol{\nabla}R\rangle$$

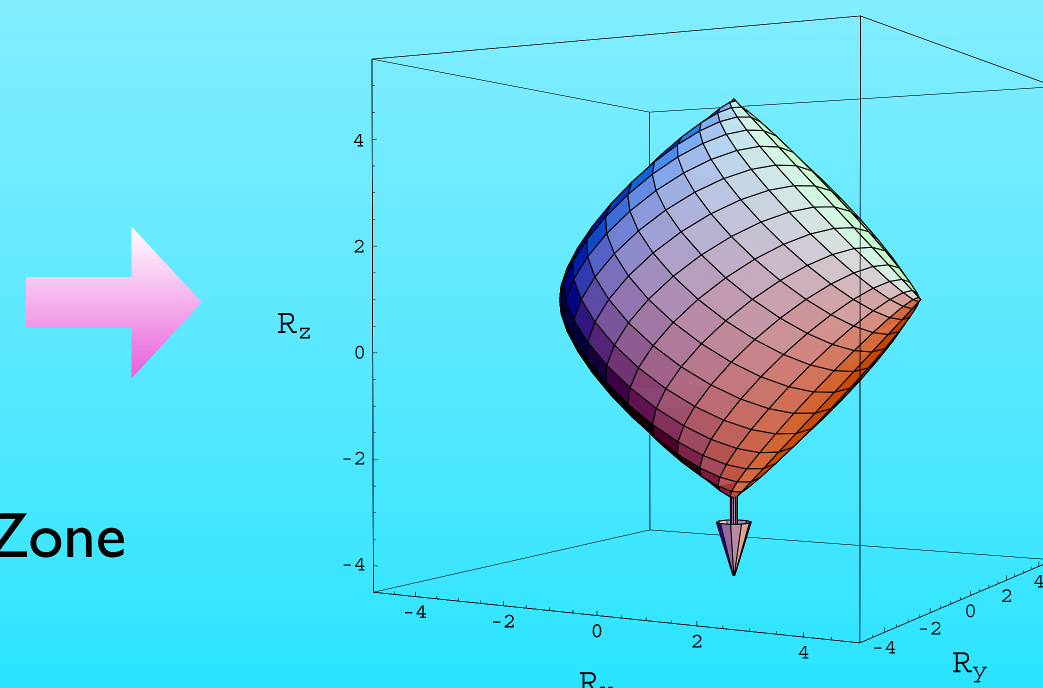
$$= -\int_{V,\,\partial V = R(BZ)} dV\,\delta(\boldsymbol{r}) = \boxed{-N_{\mathrm{covering}}}$$



$\boldsymbol{B}_R$: Monopole at the Origin



Brillouin Zone



Applications of Topological characterization of Superconductivity

- Boundary Effects: Zero Bias Conductance Peaks

- Characterization of Gap Nodes in 3D superconductivity

Boundary Effects of Superconductivity :Topological Origin

- Zero Bias Conductance Peak (ZBCP) in Anisotropic Superconductivity

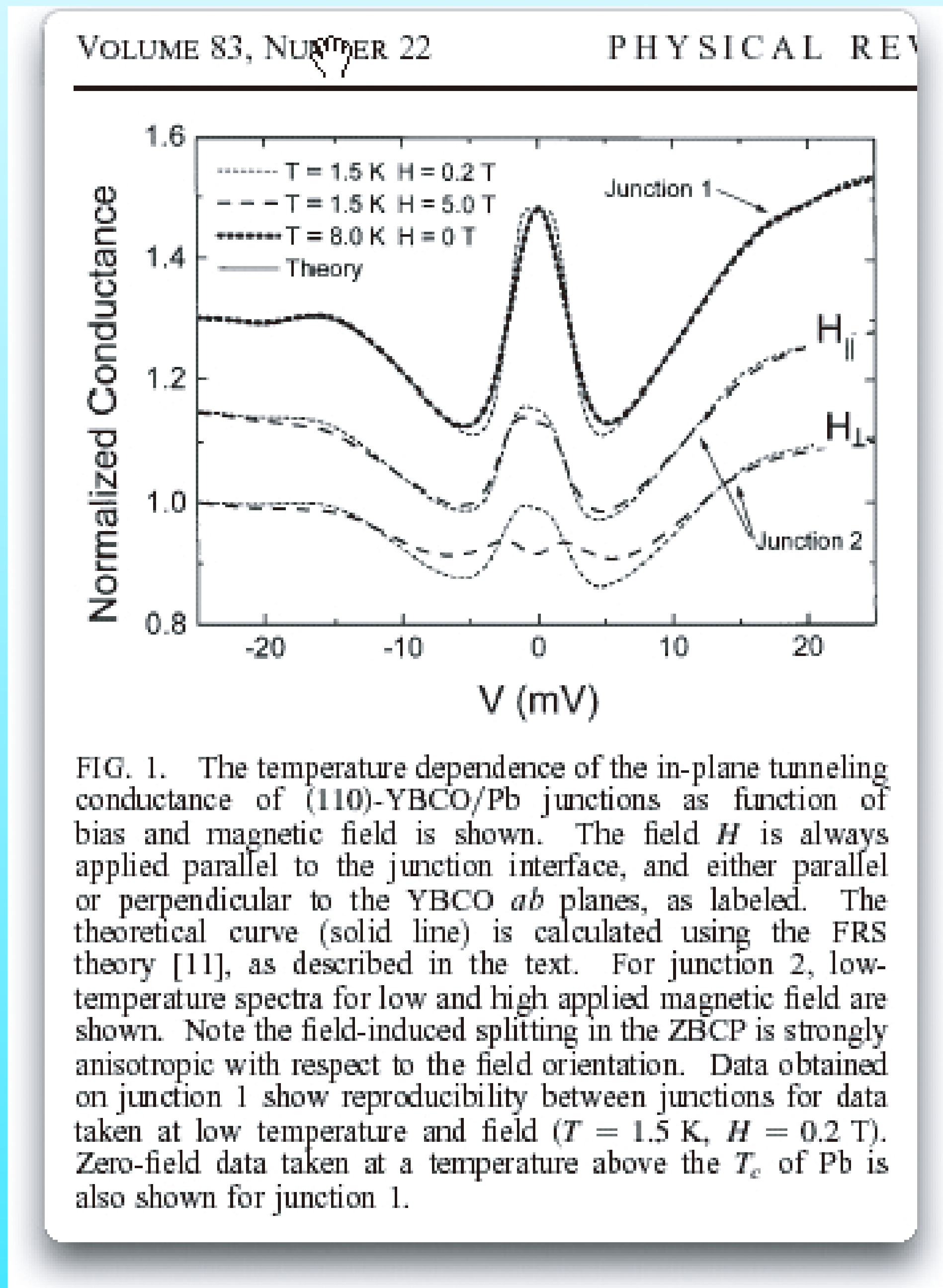

VOLUME 83, NUMBER 22 PHYSICAL REV

FIG. 1. The temperature dependence of the in-plane tunneling conductance of (110)-YBCO/Pb junctions as function of bias and magnetic field is shown. The field H is always applied parallel to the junction interface, and either parallel or perpendicular to the YBCO ab planes, as labeled. The theoretical curve (solid line) is calculated using the FRS theory [11], as described in the text. For junction 2, low-temperature spectra for low and high applied magnetic field are shown. Note the field-induced splitting in the ZBCP is strongly anisotropic with respect to the field orientation. Data obtained on junction 1 show reproducibility between junctions for data taken at low temperature and field ($T = 1.5$ K, $H = 0.2$ T). Zero-field data taken at a temperature above the T_c of Pb is also shown for junction 1.

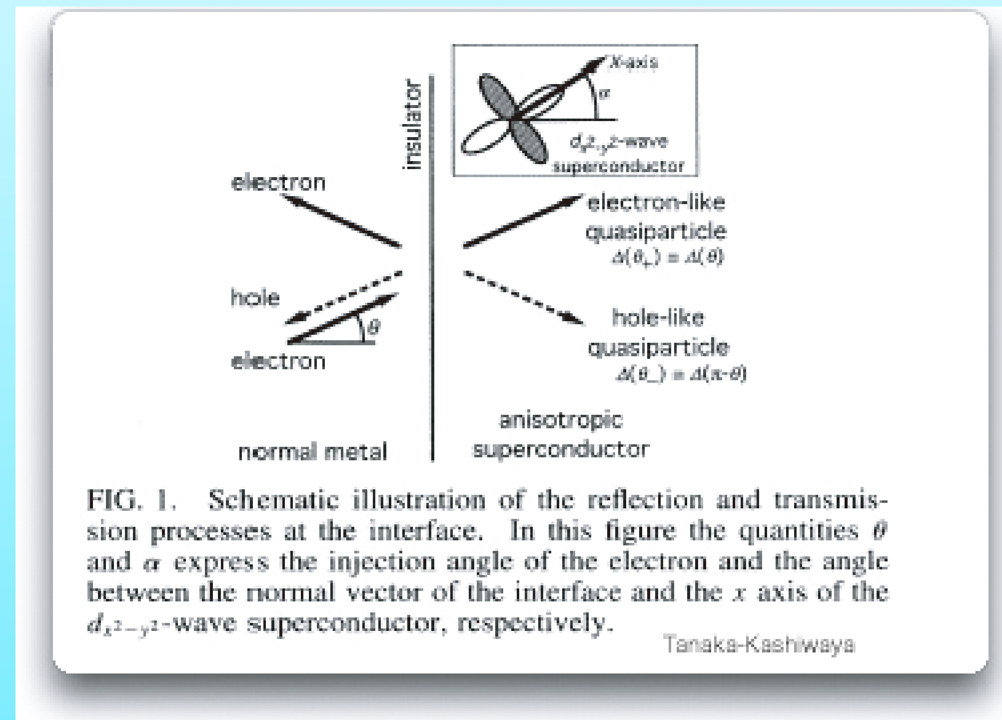


FIG. 1. Schematic illustration of the reflection and transmission processes at the interface. In this figure the quantities θ and α express the injection angle of the electron and the angle between the normal vector of the interface and the x axis of the $d_{x^2-y^2}$-wave superconductor, respectively.

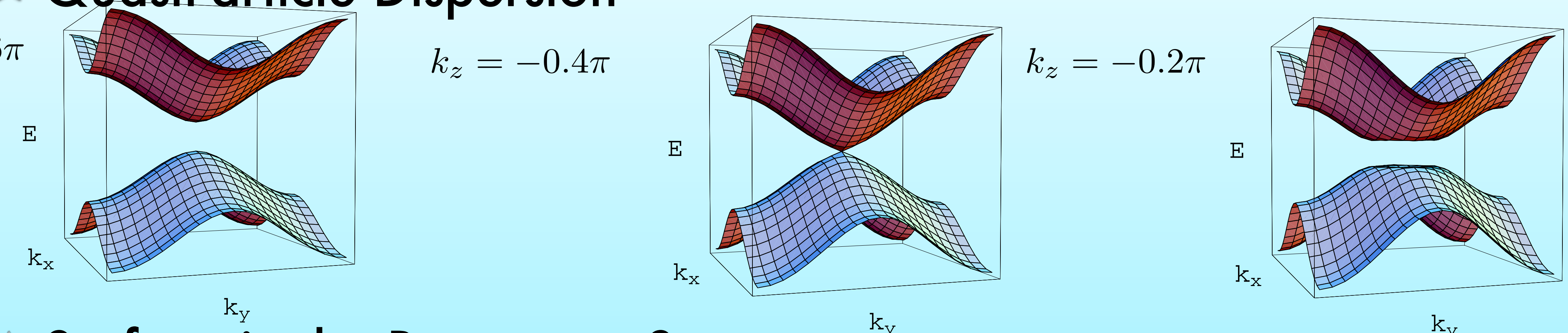
L. J. Buchholtz,G. Zwicknagl, Phys. Rev. B 23, 5788 (1981) (p-wave)
C.-R. Hu, Phys. Rev. Lett. 72, 1526 (1994) (d wave)
S. Kashiwaya, Y. Tanaka, Phys. Rev. Lett. 72, 1526 (1994)
M. Matsumoto and H. Shiba, JPSJ, 1703 (1995)
M. Aprili, E. Badica, and L. H. Greene,Phys. Rev. Lett. 83, 4630 (1999)

Topological Origin of Zero energy Edge States, Ryu-Hatsugai, PRL, 89, 077002 (2002)

Anderson-Brinkman-Morel (ABM) State

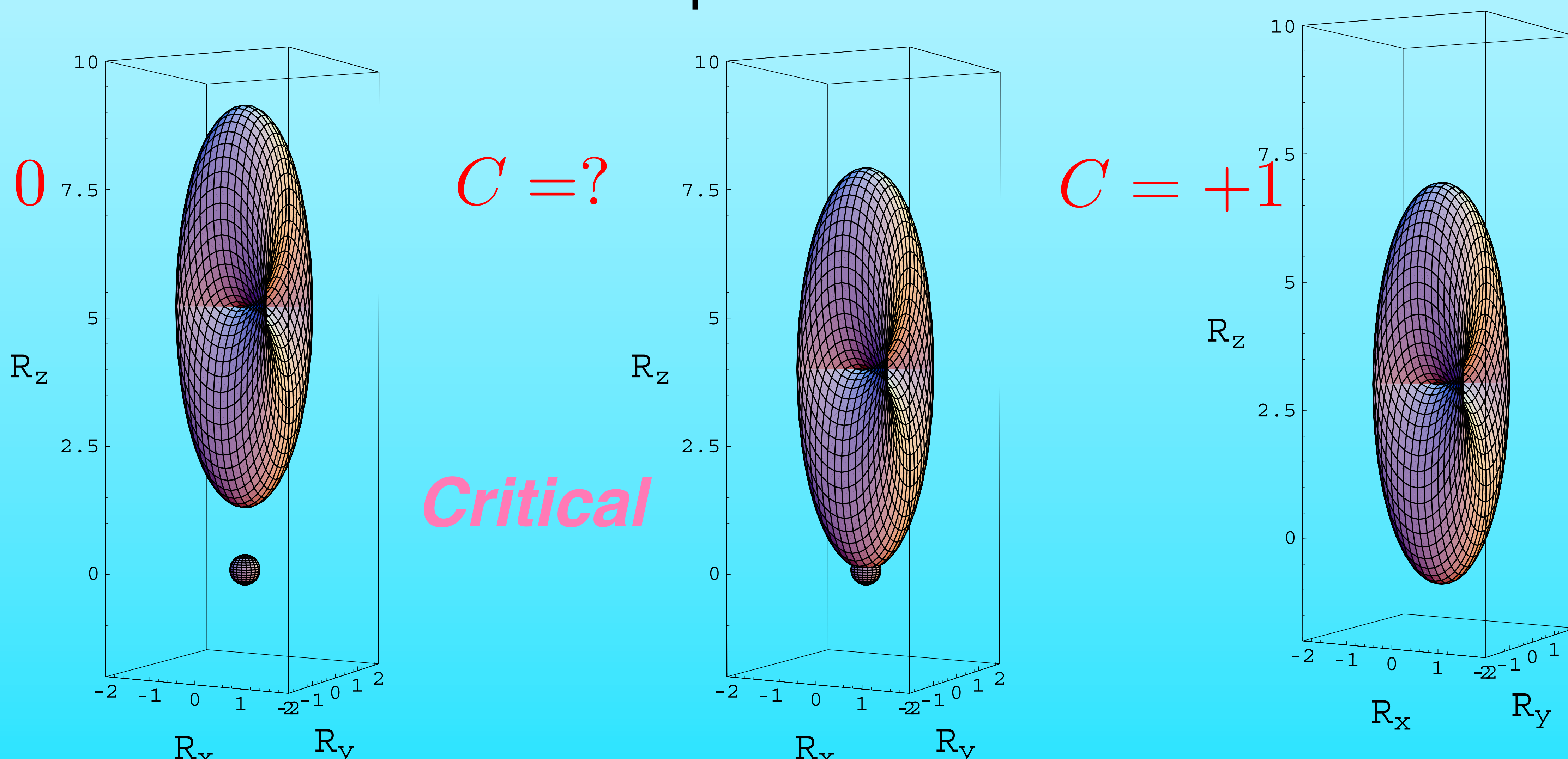
- QuasiParticle Dispersion

$k_z = -0.6\pi$ $k_z = -0.4\pi$ $k_z = -0.2\pi$



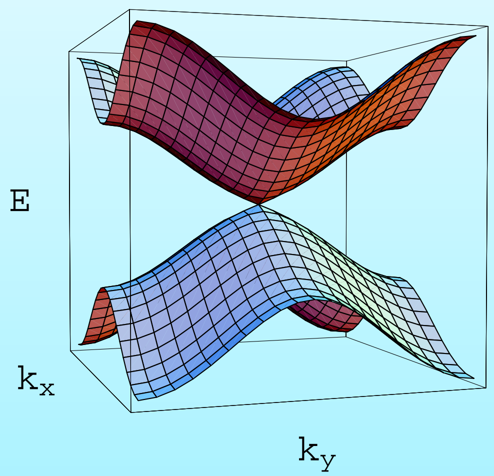
- Surface in the Parameter Space

$C = 0$ $C = ?$ $C = +1$



Critical

Gap Node as a Quantum Phase Transition



Topological Quantum Phase Transition of 2D sliced systems

with the third momentum as a parameter

↓

The Chern Number Changes

↓

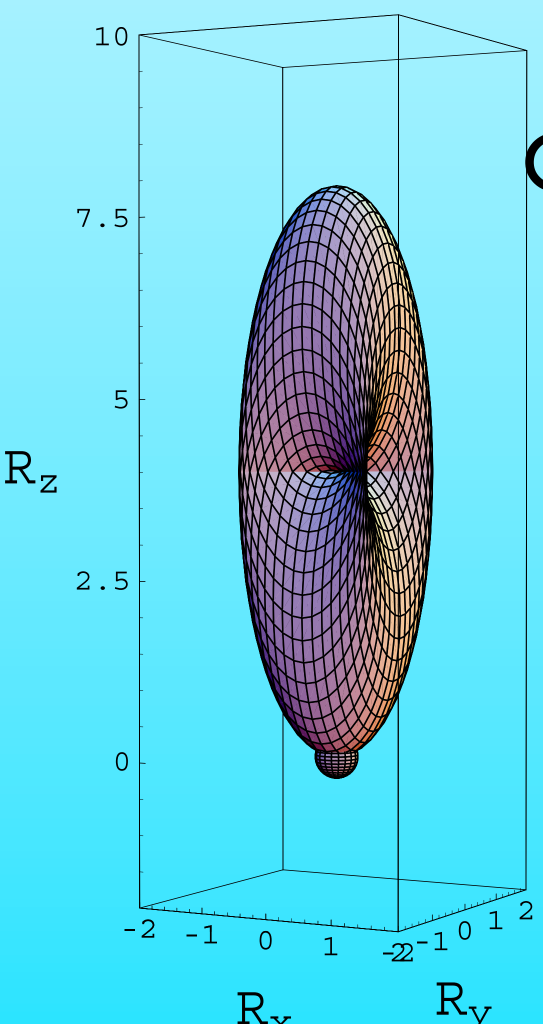
Gap Node Appears Due to the Topological Constraint

↓

Point Nodes are Generic

Line Node : Due to additional Symmetry

(Time reversal)



Generic BCS Hamiltonian on Lattice

(ref. Sigrist-Ueda, k-space)

☆ Manybody Hamiltonian

$$\boxed{H = H[\{c, c^\dagger\}] = \sum_{ij} t_{ij} c^\dagger_{i\sigma} c_{j\sigma} + \sum_{ij} V_{ij}^{\sigma_1\sigma_2;\sigma_3\sigma_4} c^\dagger_{i\sigma_1} c^\dagger_{j\sigma_2} c_{j\sigma_3} c_{i\sigma_4}}$$

$$\text{Hermiticity} \quad \to t_{ij} = t^*_{ji}, V_{ij}^{\sigma_1\sigma_2;\sigma_3\sigma_4} = (V_{ij}^{\sigma_4\sigma_3;\sigma_2\sigma_1})^*$$

$$\text{Symmetrization} \quad \to V_{ij}^{\sigma_1\sigma_2;\sigma_3\sigma_4} = V_{ji}^{\sigma_2\sigma_1;\sigma_4\sigma_3}$$

$$SU(2) \text{ invariance}: \quad H' = H[\{c', c'^\dagger\}] = H, \quad c'_{i\sigma} = U^{\sigma\sigma'} c_{i\sigma'}$$

$$\boldsymbol{U}^\dagger \boldsymbol{U} = 1_2, \qquad \det \boldsymbol{U} = 1$$

☆ Mean Field Hamiltonian with Pairing Amplitude

$$H \approx \mathcal{H} + E_0$$

$$\boxed{\mathcal{H} = \sum_{ij} t_{ij} c^\dagger_{i\sigma} c_{j\sigma} + \sum_{ij} \left(\Delta_{ij}^{\sigma_4\sigma_3 *} c_{j\sigma_3} c_{i\sigma_4} + \Delta_{ij}^{\sigma_1\sigma_2} c^\dagger_{i\sigma_1} c^\dagger_{j\sigma_2} \right)}$$

$$E_0 = -\sum_{ij} \mathrm{Tr}_{\sigma\sigma'} \boldsymbol{\Delta}^\dagger_{ij} \boldsymbol{V}^{-1} \boldsymbol{\Delta}_{ij}$$

$$\boxed{\Delta_{ij}^{\sigma_1\sigma_2} = V_{ij}^{\sigma_1\sigma_2;\sigma_3\sigma_4} \langle c_{j\sigma_3} c_{i\sigma_4} \rangle}$$

$$\frac{\partial F}{\partial \Delta_{ij}^{\sigma_4\sigma_3 *}} = 0, \qquad e^{-\beta F} = \mathrm{Tr}\, e^{-\beta(\mathcal{H} + E_0)}$$

Symmetries of Order Parameters

- Parity and Rotational Symmetry in Spin Space

$$\Delta_{ij}^{\sigma_1\sigma_2} = -\Delta_{ji}^{\sigma_2\sigma_1}$$

Fermion anticommutations

$$\text{Parity Even: singlet} \qquad \Delta_{ij}^{\sigma\sigma'} = -\Delta_{ij}^{\sigma'\sigma} = \Delta_{ji}^{\sigma\sigma'}$$

$$\text{Parity Odd : triplet} \qquad \Delta_{ij}^{\sigma\sigma'} = \Delta_{ij}^{\sigma'\sigma} = -\Delta_{ji}^{\sigma\sigma'}$$

- Matrix Notations

$$\text{singlet} \qquad \mathbf{\Delta}_{ij} = -\tilde{\mathbf{\Delta}}_{ij} = \begin{pmatrix} & \psi_{ij} \\ -\psi_{ij} & \end{pmatrix} = \psi_{ij} i\sigma_y$$

$$\psi_{ij} = \psi_{ji} \quad \text{(even)}$$

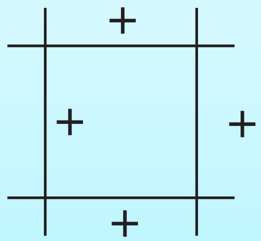
$$\text{triplet} \qquad \mathbf{\Delta}_{ij} = \tilde{\mathbf{\Delta}}_{ij} = \begin{pmatrix} -d_{ij}^x + id_{ij}^y & d_{ij}^z \\ d_{ij}^z & d_{ij}^x + id_{ij}^y \end{pmatrix} = (\vec{d}_{ij} \cdot \vec{\sigma}) i\sigma_y$$

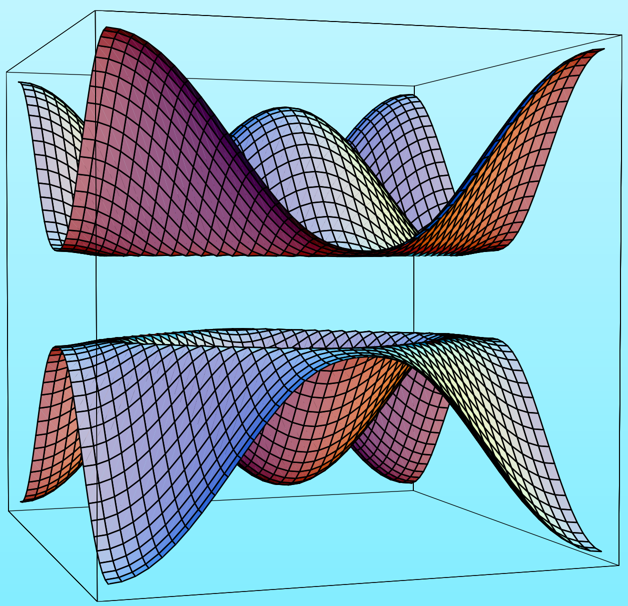
$$\vec{d}_{ij} = -\vec{d}_{ji} \quad \text{(odd)}$$

Ex. Order Parameters and Q.P. Dispersions

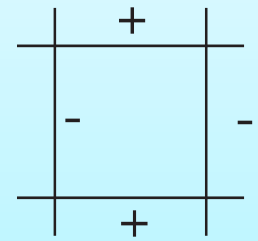
☆ 2D Examples (Singlet)

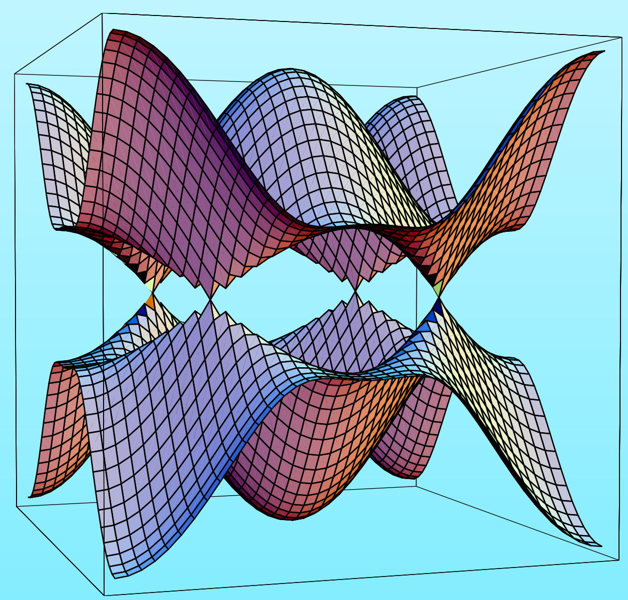
s-wave



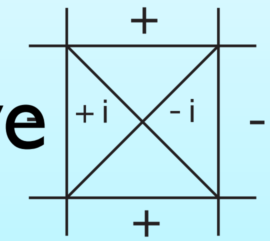


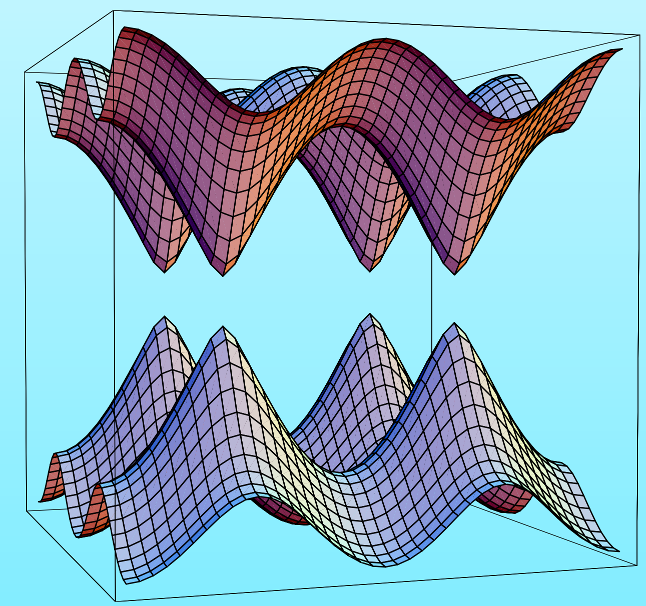
d-wave



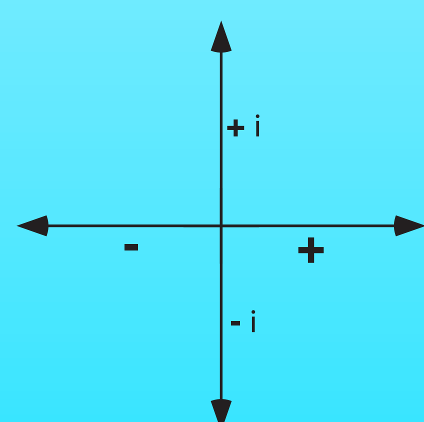


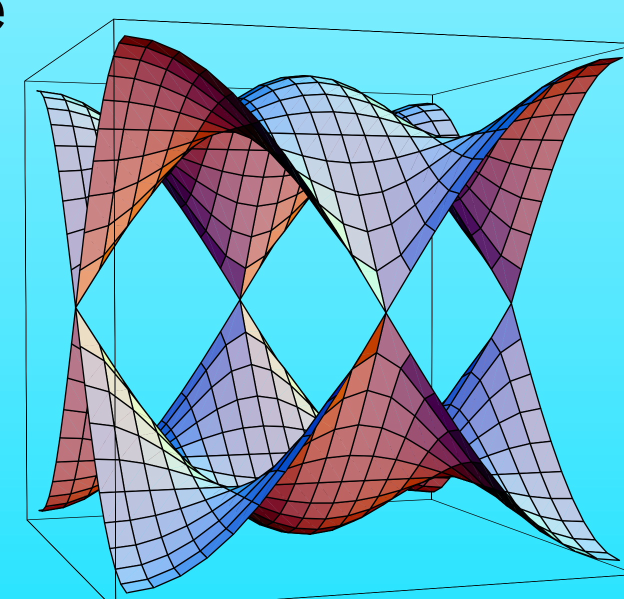
d+id-wave





☆ 2D Examples (Triplet) : Chiral p wave





Generic BCS Hamiltoniann: 4 Component Spinors

- ☆ With Translation Symmetry $$\mathcal{H} = \sum_k{}' \boldsymbol{c}(\vec{k})\boldsymbol{h}(\vec{k})\boldsymbol{c}(\vec{k}) + const.$$

$$\boldsymbol{h}(\vec{k}) = \begin{pmatrix} \epsilon(\vec{k}) & \boldsymbol{\Delta}(\vec{k}) \\ \boldsymbol{\Delta}^\dagger(\vec{k}) & -\epsilon(\vec{k}) \end{pmatrix}$$

(4 × 4 for each k)

$$\epsilon(\vec{k}) = \epsilon(-\vec{k}) : \quad t_{ij} : \mathrm{real}$$

$$\boldsymbol{c}^\dagger(\vec{k}) = (c_\uparrow^\dagger(\vec{k}), c_\downarrow^\dagger(\vec{k}), c_\uparrow(-\vec{k}), c_\downarrow(-\vec{k}))$$

- ☆ Order Parameters in Momentum Space

$$\boldsymbol{\Delta}(-\vec{k}) = -\tilde{\boldsymbol{\Delta}}(\vec{k})$$

Fermion anticommutations

$$\mathrm{singlet} \quad \boldsymbol{\Delta}(\vec{k}) = \psi(\vec{k}) i\sigma_y$$

$$\psi(-\vec{k}) = \psi(\vec{k}), \quad \tilde{\boldsymbol{\Delta}} = -\boldsymbol{\Delta}$$

$$\mathrm{triplet} \quad \boldsymbol{\Delta}(\vec{k}) = (\vec{d}(\vec{k}) \cdot \vec{\sigma}) i\sigma_y$$

$$\vec{d}(-\vec{k}) = -\vec{d}(\vec{k}), \quad \tilde{\boldsymbol{\Delta}} = \boldsymbol{\Delta}$$

Particle-Hole Symmetry & Unitarity

- Particle Hole Symmetry $$\boldsymbol{h}\begin{pmatrix}\boldsymbol{u}\\ \boldsymbol{v}\end{pmatrix}=\begin{pmatrix}\epsilon & \boldsymbol{\Delta}\\ \boldsymbol{\Delta}^\dagger & -\epsilon\end{pmatrix}\begin{pmatrix}\boldsymbol{u}\\ \boldsymbol{v}\end{pmatrix}=E\begin{pmatrix}\boldsymbol{u}\\ \boldsymbol{v}\end{pmatrix}$$
  - For Singlet Order
$$E:\begin{pmatrix}\boldsymbol{u}\\ \boldsymbol{v}\end{pmatrix}\rightleftarrows -E:\begin{pmatrix}\boldsymbol{v}^*\\ \boldsymbol{u}^*\end{pmatrix}=(\rho_x\otimes 1_\sigma)K\begin{pmatrix}\boldsymbol{u}\\ \boldsymbol{v}\end{pmatrix}$$
  - For Triplet Order
$$E:\begin{pmatrix}\boldsymbol{u}\\ \boldsymbol{v}\end{pmatrix}\rightleftarrows -E:\begin{pmatrix}-\boldsymbol{v}^*\\ \boldsymbol{u}^*\end{pmatrix}=-i(\rho_y\otimes 1_\sigma)K\begin{pmatrix}\boldsymbol{u}\\ \boldsymbol{v}\end{pmatrix}$$

- Consider $\boldsymbol{h}^2$! $$\boldsymbol{h}^2=\boldsymbol{\epsilon}^2+\begin{pmatrix}\boldsymbol{\Delta}\boldsymbol{\Delta}^\dagger & 0\\ 0 & \boldsymbol{\Delta}^\dagger\boldsymbol{\Delta}\end{pmatrix}$$

$$\boldsymbol{\Delta}\boldsymbol{\Delta}^\dagger=(\boldsymbol{\Delta}^\dagger\boldsymbol{\Delta})^*\ \&\ \text{hermite}$$
$$=\begin{cases}|\psi|^2 & :\ \text{singlet}\\ |\vec{d}|^2+\vec{\sigma}\cdot\vec{q} & :\ \text{triplet}\quad \vec{q}=i\vec{d}\times\vec{d}^*\end{cases}$$

$$E=\pm\sqrt{\epsilon^2+|\vec{d}|^2\pm q}$$

$$(\vec{\sigma}\cdot\vec{q})\boldsymbol{u}_\pm=\pm q\boldsymbol{u}_\pm$$
$$\boldsymbol{v}_\pm=-i\sigma_y\boldsymbol{u}_\mp$$
$$q=0\quad:\ \text{Unitary}$$
$$q\neq 0\quad:\ \text{Non Unitary}$$

Non-Abelian Gauge Structures for the Unitary Case

- Quasiparticles are doubly degenerate
  - Define Non-Abelian Connection (Wilczek-Zee)

$$|\boldsymbol{k}\alpha\rangle = |\alpha\rangle, \quad \alpha = 1, \cdots, M = 2$$

$$A_i^{\alpha\beta} = \langle\alpha|\partial_i|\beta\rangle, \quad \mathcal{A}^{\alpha\beta} = A_i^{\alpha\beta} dk_i, \quad \partial_i = \partial_{k_i}, i = x, y, z, \qquad (\boldsymbol{\mathcal{A}})^{\alpha\beta} \equiv \mathcal{A}^{\alpha\beta},$$

$$\boldsymbol{\mathcal{F}} \equiv d\boldsymbol{\mathcal{A}} + \boldsymbol{\mathcal{A}} \wedge \boldsymbol{\mathcal{A}} = \frac{1}{2!} \boldsymbol{F}_{ij} dk_i \wedge dk_j, \quad \boldsymbol{F}_{ij} = \partial_i \boldsymbol{A}_j - \partial_j \boldsymbol{A}_i + [\boldsymbol{A}_i, \boldsymbol{A}_j]$$

  - Base Change in the degenerate space: $|\alpha\rangle \to |\bar{\alpha}\rangle = |\alpha\rangle \omega^{\alpha\bar{\alpha}}, \quad \boldsymbol{\omega} : \text{unitary}$
    - Fixing a phase is not enough to uniquely define the Connections

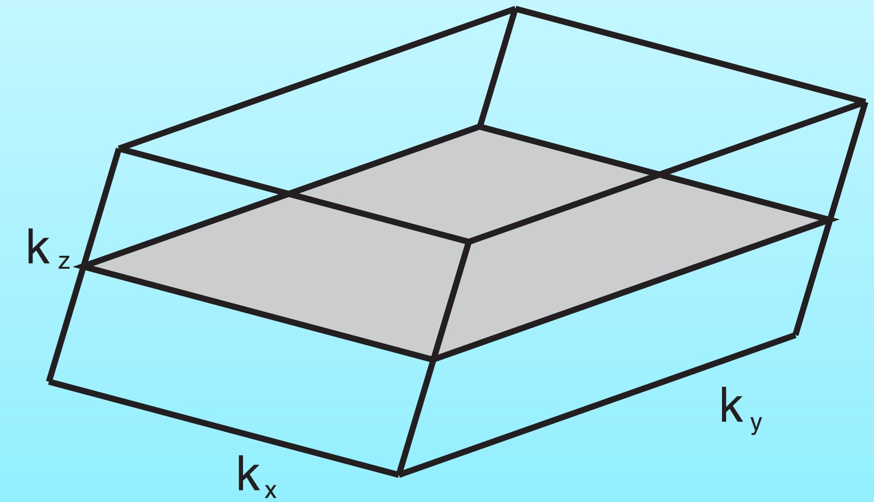
$$\bar{\boldsymbol{\mathcal{A}}} = \boldsymbol{\omega}^\dagger \boldsymbol{\mathcal{A}} \boldsymbol{\omega} + \boldsymbol{\omega}^\dagger d\boldsymbol{\omega}, \qquad \bar{\boldsymbol{\mathcal{F}}} = \boldsymbol{\omega}^\dagger \boldsymbol{\mathcal{F}} \boldsymbol{\omega}$$

$$\mathrm{Tr}\ \bar{\boldsymbol{\mathcal{F}}} = \mathrm{Tr}\ \boldsymbol{\mathcal{F}} : \quad \text{Invariant}$$

Topological Invariants for the Unitary Case

- The First Chern Number: Chiral Anomaly

$$C_i(k_i) = \frac{1}{2\cdot 2!}\epsilon_{ijk}\frac{1}{2\pi i}\int_{T^2_{jk}} \mathrm{Tr}\,\boldsymbol{\mathcal{F}}$$



$$C_x(k_x) = \frac{1}{2\pi i}\int_{T^2_{yz}} dk_y dk_z\ B_x$$

$$C_y(k_y) = \frac{1}{2\pi i}\int_{T^2_{zx}} dk_z dk_x\ B_y$$

$$C_z(k_z) = \frac{1}{2\pi i}\int_{T^2_{xy}} dk_x dk_y\ B_z$$

3D = 2D Cross Sections with the third dimension as a parameter

$$B_x = \mathrm{Tr}\,\boldsymbol{F}_{yz}/2, B_y = \mathrm{Tr}\,\boldsymbol{F}_{zx}/2, B_z = \mathrm{Tr}\,\boldsymbol{F}_{xy}/2$$

Chern Numbers for the Unitary Order

☆ Gauge Fixing: Bases in Degenerate

$$\boldsymbol{\Delta}^{\dagger}\boldsymbol{\Delta} = |\Delta|^2 1_2, \quad \boldsymbol{\Delta} = |\Delta|\boldsymbol{\Delta}_0, \quad \boldsymbol{\Delta}_0 : \text{Unitary} \qquad \begin{aligned}\boldsymbol{\Delta}_0|1\rangle &= e^{-i\phi_1}|1\rangle \\ \boldsymbol{\Delta}_0|2\rangle &= e^{-i\phi_2}|2\rangle\end{aligned}$$

☆ Reduction from 4 × 4 to 2 × 2 (Without ESP

$$\begin{pmatrix} \epsilon 1_\sigma & |\Delta|\boldsymbol{\Delta}_0 \\ |\Delta|\boldsymbol{\Delta}_0^{-1} & -\epsilon 1_\sigma \end{pmatrix}\begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix} = E\begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix} \Rightarrow \begin{pmatrix} \epsilon & |\Delta| \\ |\Delta| & -\epsilon \end{pmatrix}_\rho \otimes 1_\sigma \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{\Delta}_0\boldsymbol{v} \end{pmatrix} = E\begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{\Delta}_0\boldsymbol{v} \end{pmatrix}$$

$$\begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{\Delta}_0\boldsymbol{v} \end{pmatrix} = |R\rangle_\rho \otimes |i\rangle_\sigma, \qquad |R\rangle_\rho = \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}$$

☆ Non-Abelian Connection

$$|\psi_\alpha\rangle = \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix}_\alpha = \begin{pmatrix} -\sin\frac{\theta}{2}|\alpha\rangle_\sigma \\ e^{i\phi_\alpha}\cos\frac{\theta}{2}|\alpha\rangle_\sigma \end{pmatrix}, \qquad \alpha = 1, 2$$

$$A_i^{\alpha\beta} = \langle\psi_\alpha|\partial_i|\psi_\beta\rangle = \langle R_\alpha|\partial_i|R_\beta\rangle_\rho\langle\alpha|\beta\rangle_\sigma + \langle R_\alpha|R_\beta\rangle_\rho\langle\alpha|\partial_i|\beta\rangle_\sigma$$

C. Numbers for the Unitary Order (cont.)

- Explicit Calculations

$$
\begin{aligned}
A_i^{\alpha\beta} =& A_i^{\alpha}(\rho)\delta_{\alpha\beta} + \langle R_\alpha | R_\beta \rangle A_i^{\alpha\beta}(\sigma) \\
A_i^{\alpha}(\rho) =& \langle R_\alpha | \partial_i | R_\alpha \rangle_\rho, \quad |R_\alpha\rangle_\rho = \begin{pmatrix} -\sin\frac{\theta}{2} \\ e^{i\phi_\alpha}\cos\frac{\theta}{2} \end{pmatrix} \\
A_i^{\alpha\beta}(\sigma) =& \langle \alpha | \partial_i | \beta \rangle_\sigma = (\boldsymbol{U}^\dagger \partial_i \boldsymbol{U})_{\alpha\beta} \\
\boldsymbol{A}_i(\sigma) =& \boldsymbol{U}^\dagger \partial_i \boldsymbol{U}, \quad \boldsymbol{U} = (|1\rangle_\sigma, |2\rangle_\sigma)
\end{aligned}
$$

- Fictitious Magnetic Field

$$
\begin{aligned}
B_i =& B_i(\rho) + B_i(\sigma) \\
B_i(\rho) =& \epsilon_{ijk} \mathrm{Tr}\, \partial_j \boldsymbol{A}_k(\rho) \\
B_i(\sigma) =& \epsilon_{ijk} \mathrm{Tr}\, \partial_j \boldsymbol{A}_k(\sigma) = 0
\end{aligned}
$$

Sum rule

$$
\begin{aligned}
B_i(\sigma) =& \epsilon_{ijk} \partial_j \mathrm{Tr}\, \boldsymbol{A}_k(\sigma) \\
=& \epsilon_{ijk} \mathrm{Tr}\, \partial_j (\boldsymbol{U}^{-1} \partial_k \boldsymbol{U}) \\
=& -\epsilon_{ijk} \mathrm{Tr}\, (\boldsymbol{U}^{-1} \partial_j \boldsymbol{U} \boldsymbol{U}^{-1}) \partial_k \boldsymbol{U}) \\
=& -\epsilon_{ijk} \mathrm{Tr}\, (\boldsymbol{U}^{-1} \partial_j \boldsymbol{U})(\boldsymbol{U}^{-1} \partial_k \boldsymbol{U}) \\
=& 0, \qquad (\partial \boldsymbol{U} = \boldsymbol{U}^{-1} \partial \boldsymbol{U} \boldsymbol{U}^{-1})
\end{aligned}
$$

Non Unitary Cases (4 by 4)

- Classification by the Helicity $\vec{q} = i\vec{d} \times \vec{d}^*$ $\qquad \not{q}\, \boldsymbol{u}_\pm = \pm q\, \boldsymbol{u}_\pm$
- Reduction to traceless 2 by 2 Hamiltonian
  - Berry's Parametrization
  - Dirac Monopoles
  - Graphical Treatment for the Chern number

$$^\forall \not{O} = \vec{\sigma} \cdot \vec{O}$$

helicity: $+q$

$$H^+_{eff} = \not{R}_+$$

$$\vec{R}_+ = (\epsilon, \mathrm{Im}\,\langle \not{d} \rangle_{+-}, \mathrm{Re}\,\langle \not{d} \rangle_{+-}),$$

$$E_+ = \pm\sqrt{\epsilon^2 + |\vec{d}|^2 + q} = \pm|\vec{R}_+|$$

helicity: $-q$

$$H^-_{eff} = \not{R}_-$$

$$\vec{R}_- = (\epsilon, -\mathrm{Im}\,\langle \not{d}^* \rangle_{+-}, \mathrm{Re}\,\langle \not{d}^* \rangle_{+-}),$$

$$E_- = \pm\sqrt{\epsilon^2 + |\vec{d}|^2 - q} = \pm|\vec{R}_-|$$