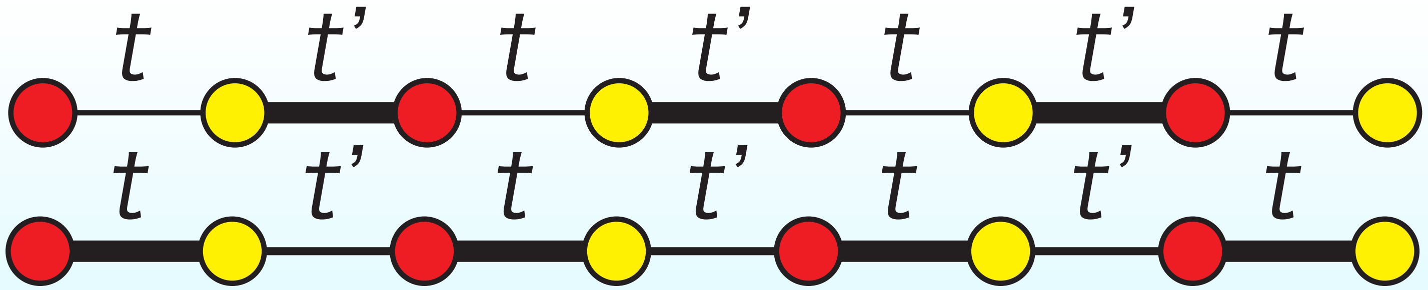
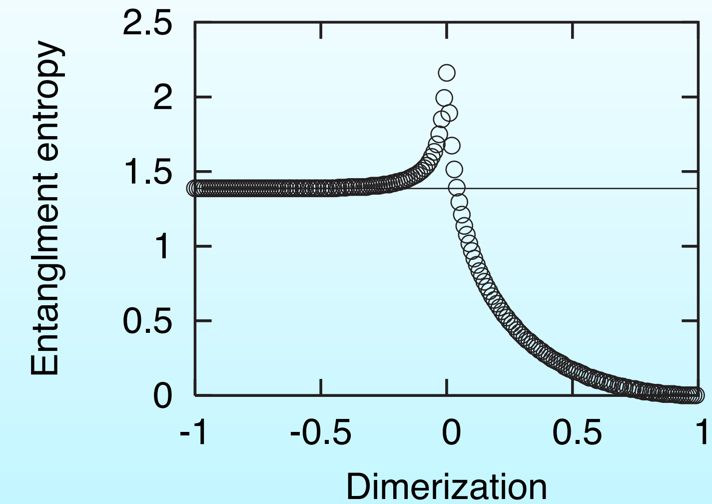
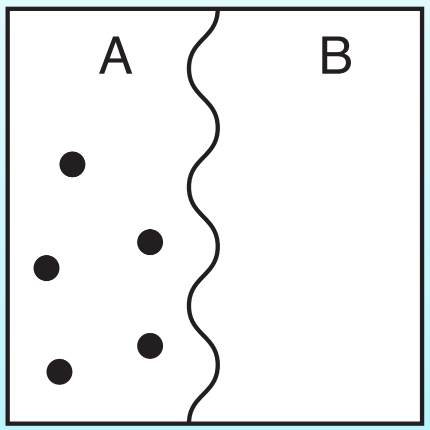


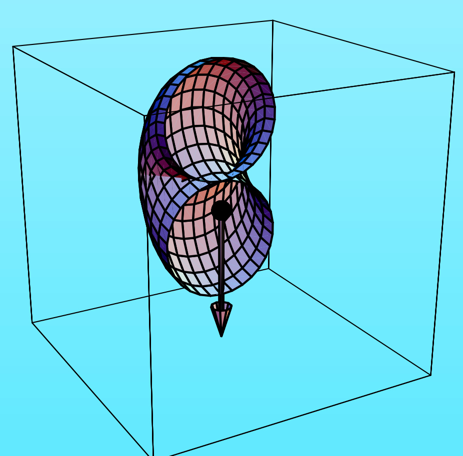
JPSJ Fall meeting 2005





Entanglement Entropy and Berry’s Phases

One way To Characterize Topological Order



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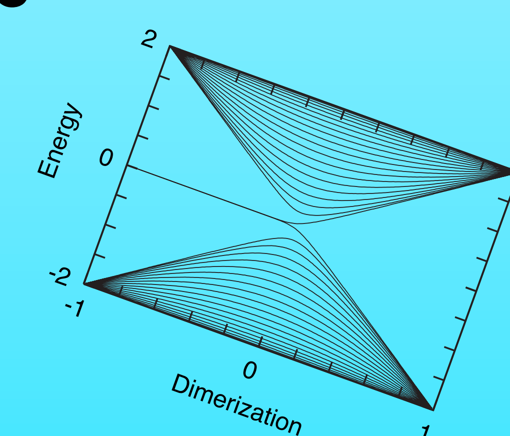


Ref. S.Ryu and Y. Hatsugai, Phys. Rev. B73, 245115 (2006)

Plan of the Talk

- Topological Orders
  - New kinds of Order for Quantum State
  - Need Physical Quantities without Any Symmetry Breaking
- Entanglement Entropy
  - What is it ? Entropy ?
  - Mixed State and Entanglement
- Edge States and Entanglement Entropy
  - Generic Formulation (without Interaction)
  - Zero Modes by Chiral Symmetry
  - Berry's Phases
- Applications
  - One dimensional : Dimmer model (Su-Schrieffer-Heeger, SSH)
  - Two dimensional : p-wave

What is a Topological Order?

- *Try to Characterize Quantum Mechanical states Without Any Symmetry Breaking*
  - Topological Degeneracy
  - Gauge Structures naturally appear
- *Use Geometrical Phases*
  - Allow Degeneracies
  - Generic Energy Gap: Topological Insulators
- *Topological Numbers as Topological Order Parameters*
  - The (first) Chern Numbers
  - Gauge Invariants

Y.H, cond-mat/0405551, J. Phys. Soc. Jpn.73, 2604 (2004)
cond-mat/0412344, J. Phys. Soc. Jpn.74, 1374 (2005)

- *Use Entanglement Entropy!*

Topological Orders: Useful !

- *Quantum Hall Effects*
- *Topological Equivalence* **between** *Anisotropic Superconductivity* *and* *Carbon 2D systems*
  - Zero Bias Peaks - Boundary Local Moments
  - Pierles Instabilities of ZERO Energy Edge States (Protected by Symmetry)

[1] S.R & Y.H., Phys. Rev.Lett. 89, 077002(2002)

[2] S.R & Y.H., Physics C 388-389, 78 (2003)

[3] S.R & Y.H., Phys. Rev. B67, 165410 (2003)

[4] S.R & Y.H., Physica E 22, 679 (2004)

- *Polarization* **of Insulators,** *KSV* **formula**
- *Haldane Spin Chain*
- Generic *Quantum Liquids*

Y.H, J. Phys. Soc. Jpn.74, 1374 (2005)

Entropy in Quantum Physics

Mixed States and Pure States

ρ : density mat.

$$S = -\langle \log \rho \rangle = -\mathrm{Tr}\, \rho \log \rho : \ \text{Entropy}$$

$$\rho_{pure} = |\Psi\rangle\langle\Psi|$$

$$\rho_{mix} = \sum_j p_j |\Psi_j\rangle\langle\Psi_j|$$

$$\langle \Psi_j | \Psi_k \rangle = \delta_{jk}$$

$$\sum_j p_j = 1, p_j \geq 0$$

ex. $p_j \propto e^{-\beta E_j}$

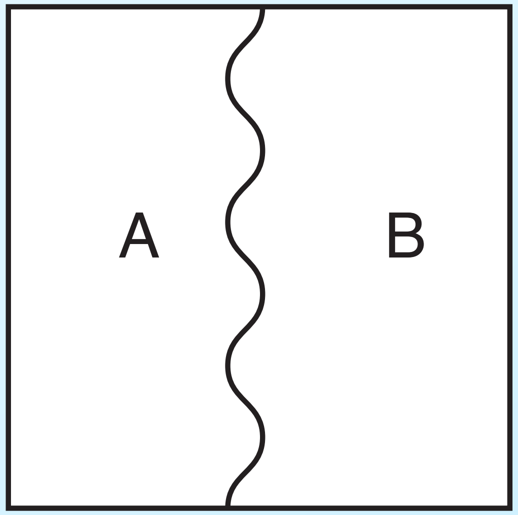
$$S_{pure} = 0$$

$$S_{mix} = -\sum_j p_j \log p_j > 0$$

Entanglement Entropy

System $= A \oplus B$

State $= \sum \Psi_A \otimes \Psi_B$



☆ Mixed State From Entanglement

☆ Direct Product State

$$|\Psi_{AB}\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle$$

☆ Entangled State

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{D}} \sum_j^D |\Psi_A^j\rangle \otimes |\Psi_B^j\rangle$$

☆ Partial Trace

$$\rho_{AB} = |\Psi_{AB}\rangle\langle\Psi_{AB}|$$

$$\rho_A = \mathrm{Tr}_B \, \rho_{AB}$$

$$= \frac{1}{D} \sum_j^D |\Psi_A^j\rangle\langle\Psi_A^j|$$

Pure State

$D = 1$

Mixed State

$$\rho_{AB} = \frac{1}{D} \sum_{jk}^D |\Psi_A^j\rangle\langle\Psi_A^k| \otimes |\Psi_B^j\rangle\langle\Psi_B^k|$$

☆ How much is the State Entangled ?

Entanglement Entropy :

$$\boxed{S_A = -\langle \log \rho_A \rangle = \log D}$$

Whtat's obtained in Related Topics

- *Entanglement Entropy in Quantum Information*
  - Osterloh,Amito,Falci and Fazio, Nature 416 (2002)
  - Osborne and Nielsen, Phys. Rev.A 66 032110 (2002)
  - Vidal, Latorre, Rico and Kitaev, PRL. 90, 227902 (2003)
  - Holzhey, Larsen and Wilczek, Nucl. Phys. B424 443, (1994)

Entropy of Black Holes

- *New Tools for Quantum Criticality*
  - Calabrese and Cardy, hep-th/045152
  - Kitaev, cond-mat/00010440
  - Paschel, cond-mat/0212631
  - Fan, Korepin and Roychowdhury, PRL. 93 2279203 (2004)

Entanglement Entropy For Fermions

- Fermi Sea and Correlations $\boldsymbol{P} = \sum_{\epsilon_\ell < 0} \boldsymbol{P}_\ell$

$$\langle c_j^\dagger c_k \rangle = (\boldsymbol{P})_{kj} \qquad \boldsymbol{h} = \sum_\ell \epsilon_\ell \boldsymbol{P}_\ell, \ \boldsymbol{P}_\ell = \boldsymbol{\psi}_\ell \boldsymbol{\psi}_\ell^\dagger, \ \boldsymbol{h}\boldsymbol{\psi}_\ell = \epsilon_\ell \boldsymbol{\psi}_\ell$$

- Entanglement Entropy <u>without Interaction</u>
  - Wick's Theorem:Two Point Functions Determine Everything!

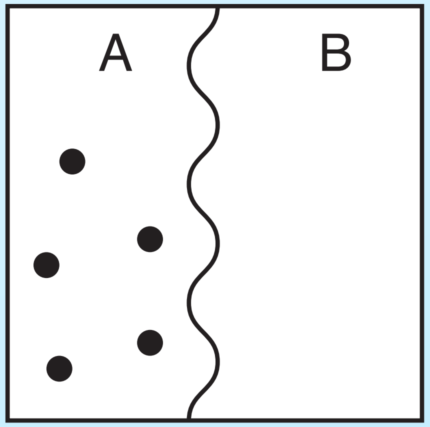
$$\langle c_{i_1}^\dagger c_{i_2}^\dagger \cdots c_{j_2} c_{j_1} \rangle = \sum_P (-)^P \langle c_{i_{P1}}^\dagger c_{j_1} \rangle \langle c_{i_{P2}}^\dagger c_{j_2} \rangle \cdots$$

  - Observables in A space are Invariant under the Partial Trace

$$\langle c_i^\dagger c_j \rangle_{AB} = \langle c_i^\dagger c_j \rangle_A \text{ if } i, j \in A$$

- Mixed States without Interactions

$$\rho_A = \frac{1}{\mathcal{N}} e^{-\mathcal{K}}, \ \mathcal{N} = \mathrm{Tr}_A\, e^{-\mathcal{K}}, \quad \mathcal{K} = \boldsymbol{c}_A^\dagger \tilde{\boldsymbol{h}} \boldsymbol{c}_A \qquad \boldsymbol{c} = \begin{bmatrix} \boldsymbol{c}_A \\ - \\ \boldsymbol{c}_B \end{bmatrix}$$



$$\tilde{\boldsymbol{h}} = \sum_\ell \tilde{\epsilon}_\ell \tilde{\boldsymbol{P}}_\ell, \ \tilde{\boldsymbol{P}}_\ell = \tilde{\boldsymbol{\psi}}_\ell \tilde{\boldsymbol{\psi}}_\ell^\dagger, \ \tilde{\boldsymbol{h}} \tilde{\boldsymbol{\psi}}_\ell = \tilde{\epsilon}_\ell \tilde{\boldsymbol{\psi}}_\ell$$

$$\langle c_j^\dagger c_k \rangle_A = \sum_\ell \zeta_\ell (\tilde{\boldsymbol{P}})_{kj}, \zeta_\ell = f(\tilde{\epsilon}_\ell), \ f(x) = \frac{1}{e^x + 1}$$

- Entanglement Entropy

ζ_ℓ : Eigen Values of the $\boldsymbol{C}$ Matrix (Correlation Matrix)

$$\langle c_i^\dagger c_j \rangle_{AB} = \langle c_i^\dagger c_j \rangle_A = (\boldsymbol{C})_{ji}$$

$$S_A = -\sum_\ell \{(1 - \zeta_\ell) \log(1 - \zeta_\ell) + \zeta_\ell \log \zeta_\ell\}$$

Recipe for Fermions without Interactions

- Construct Full Eigen States for the "AB" System
- Construct the Projection into the Fermi Sea: Correlation Matrix of the "AB" system
- Cut the System to Obtain the C matrix

Edge States !

Trace over the "B" system

$$\boldsymbol{P} = \sum_{\epsilon_\ell < 0} \boldsymbol{P}_\ell = \sum_{\epsilon_\ell < 0} \boxed{\psi_\ell}\,\boxed{\quad \psi_\ell^\dagger \quad} = \begin{array}{c|cc} & A & B \\ \hline A & C & * \\ B & * & * \end{array}$$

- Diagonalize the C Matrix : Eigenvalues ζ_ℓ
- Calculate the Entanglement Entropy from ζ_ℓ

Edge States and Entanglement Entropy

☆ Correlation Matrix as a Fictitious Hamiltonian

- ☆ Original Correlation Matrix $\boldsymbol{P}$: Eigen Values= 1,0 *(Projection)*
  - ☆ Eigen States are the same as those of the Hamiltonian $\boldsymbol{H}$

$$\boldsymbol{H}\Psi_j = E_j \Psi_j$$

$$\boldsymbol{P}\Psi_j = \zeta_j \Psi_j, \qquad \zeta_j = 0, 1$$

  - ☆ Two Band (Flat Band) Hamiltonian (highly degenerate)
- ☆ When the System is gapped, the Correlations are short range.
- ☆ Truncated Correlation Matrix $\boldsymbol{C}$:

Short-Range Hamiltonian with Boundaries

Possible Appearance of Edge States

with Energy between the gap [0,1].

$$\boldsymbol{C}\Psi_j = \zeta_j \Psi_j$$

$$0 < \zeta_j < 1 \ ({}^{\exists} j)$$

→ *It CAN contribute to* S_A *!*

$$S_A = -\sum_\ell \{(1-\zeta_\ell)\log(1-\zeta_\ell) + \zeta_\ell \log \zeta_\ell\}$$

Topological Origin of Edge States

☆ When and Why do the Edge States Appear ??

Topological Origins Characterized by Bulk !

<u>Bulk — Edge Correspondence</u>

2D QHE, Y.H. PRL 71, 3697 (1993)

Stable Mid-Band Edge States

Protected by Chiral Symmetry

S.R & Y.H. PRL 89, 077002 (2002)

Lower Bounds

For the Entanglement Entropy

$$^{\exists}\zeta_j^{R,L} = \frac{1}{2}, \qquad S_A \geq 2\log 2$$

1D Exactly Solvable Dimer Model

- The Hamiltonians and Correlation Matrices are diagonalized Explicitly (with Boundaries)

$$H = \sum_k \boldsymbol{c}^\dagger(k)\,[\,\boldsymbol{\sigma}\cdot\boldsymbol{R}(k)\,]\,\boldsymbol{c}(k)$$

$$\boldsymbol{R} = (-\Delta\cos k, -\Delta\sin k, \xi)$$

- Trancated Correlation Matrix C has two non trivial eigen values other than 0 and 1

$$^{\exists}\zeta_j^{R,L} = \frac{1}{2}\left(1 \pm \frac{\xi}{|\boldsymbol{R}|}\right)$$

$$S_A/2 = -\frac{\gamma}{2\pi}\log\frac{\gamma}{2\pi} - \frac{2\pi-\gamma}{2\pi}\log\frac{2\pi-\gamma}{2\pi}$$

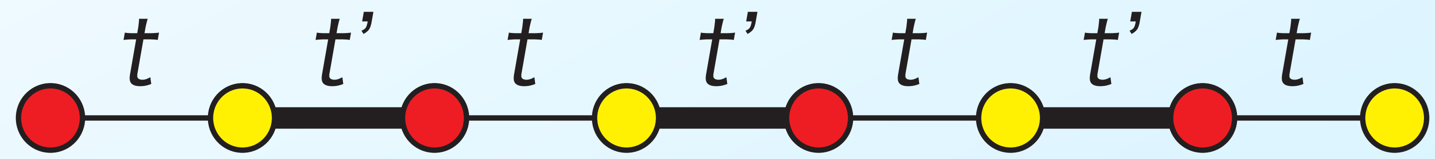
$$H|v_-\rangle = -R|v_-\rangle$$

Direct Relation $\updownarrow$

$$\gamma = i\int_0^{2\pi} dk\,\langle v_-|\frac{d}{dk}|v_-\rangle,$$

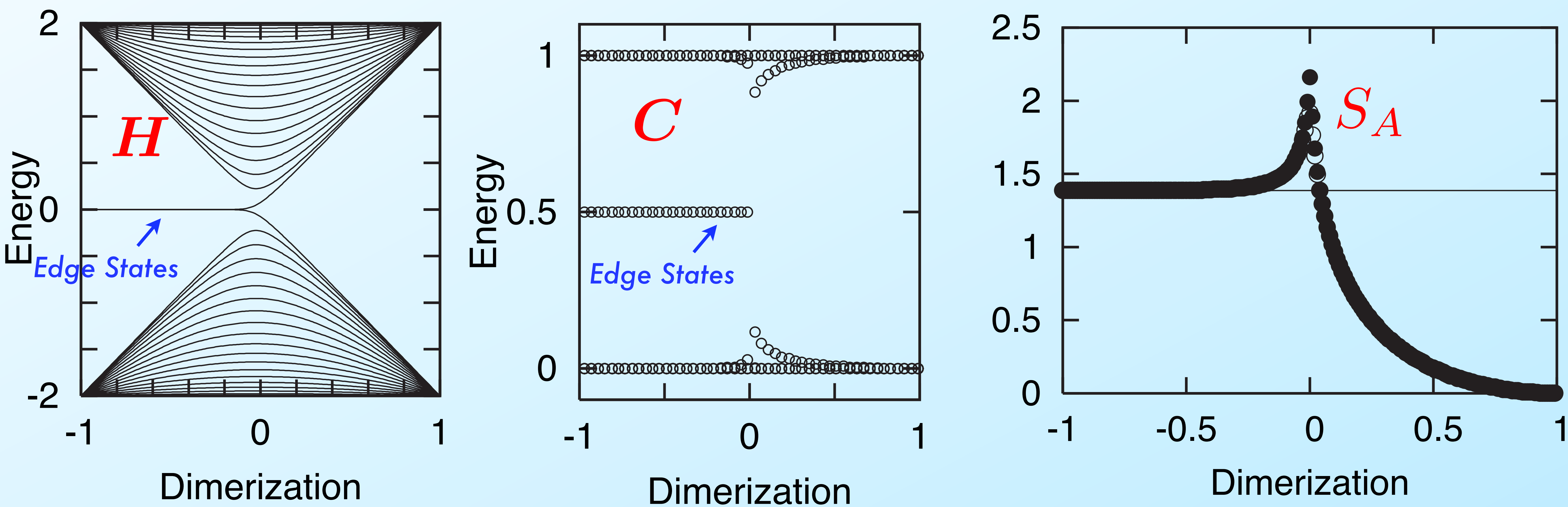
$\boxed{\text{Zak's phase}}$

SSH Model in 1 Dimension



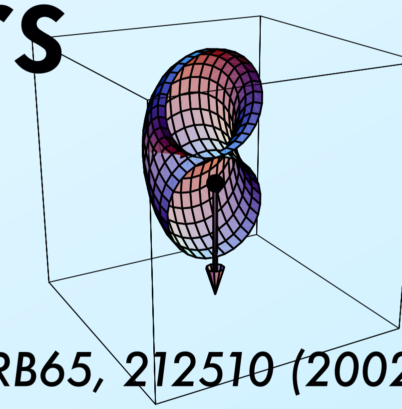
☆ Chiral Symmetry (Bipartite) Protects Mid-Energy Edge State Flat Band

Spectrum | Spec. of Correlation Mat. | Entanglement Entropy



- Symmetry Protected Edge States if $t/t' < 1$ both for $\boldsymbol{H}$ and $\boldsymbol{C}$
- Lower Bound for the Entanglement Entropy $S_A \geq 2\log 2 = 1.38\ldots$

Entanglemenet Entropy for 2D Chiral p-wave Superconductors



☆ Topological Quantum Phase Transitions Accompanied by Chern Number Changes

Y.H.& S.R.,PRB65, 212510 (2002)

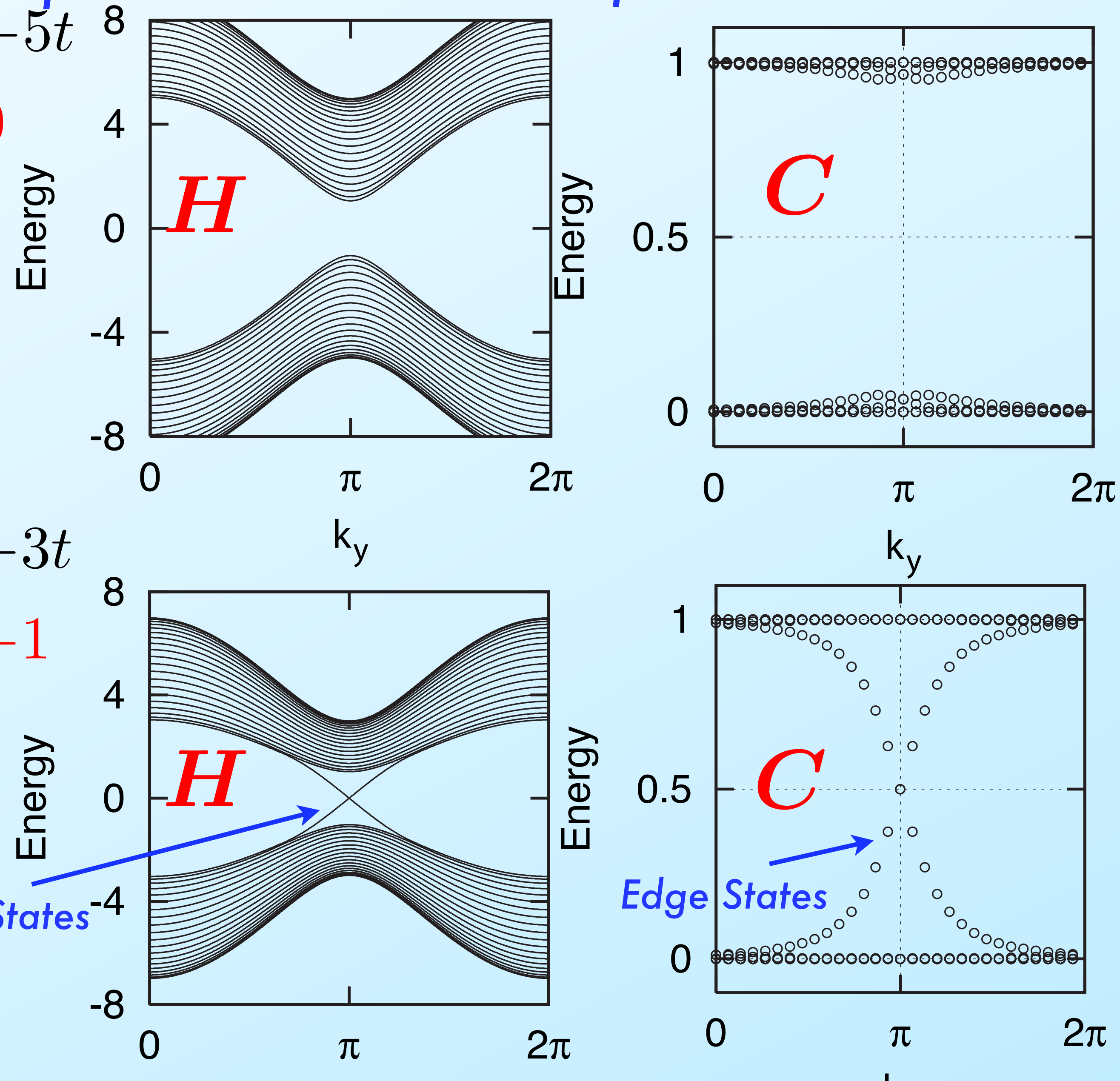
Spec. of Hamiltonian Spec. of Correlation Mat.

$\mu = -5t$

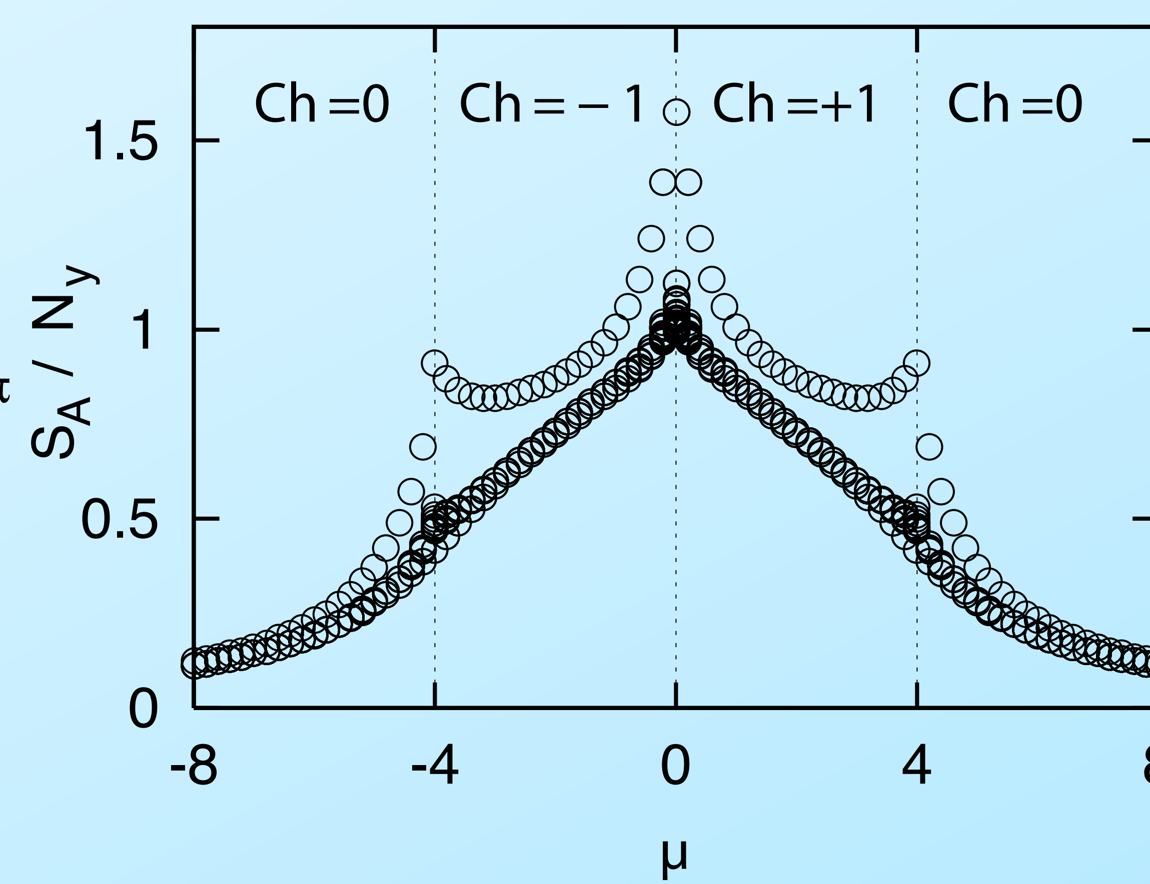
$Ch = 0$

$\mu = -3t$

$Ch = -1$



Entanglement Entropy



$t = \Delta = 1$

Summary

- *Quantum Mixed States and Entropy*
- *Mixed States from Entanglement*
- *Entanglement Entropy for Topological Nontrivial Gapped States*
- *Edge States supply Lower Bounds*
- *Useful Measure for the Topological Orders*
- *SSH Model in 1Dimension & Chiral p-wave Superconductors in 2D*

END