量子力学3 201310885 松本 祥

3次元回転

$R(\alpha,\beta,\gamma)=R_\alpha(e_z)R_\beta(e_y)R_\gamma(e_z)$　　α,β,γ : Euler angles

Wave function ψ

$R:\psi\mapsto R\psi$　　$[H,R]=0$

d重縮退　$|\psi_1\rangle,|\psi_2\rangle,\cdots,|\psi_d\rangle$

$H|\psi_i\rangle=E|\psi_i\rangle$

$H(R|\psi_i\rangle)=RH|\psi_i\rangle=E(R|\psi_i\rangle)$

$\longrightarrow R|\psi_i\rangle\in E$の固有状態

$R|\psi_i\rangle=|\psi_j\rangle D_{ji}(R)$　線型結合

$R(|\psi_1\rangle,|\psi_2\rangle,\cdots,|\psi_d\rangle)=(|\psi_1\rangle,|\psi_2\rangle,\cdots,|\psi_d\rangle)D(R)$

$R\psi=\psi D(R)$　　$D(R)$: $d\times d$行列　$\psi=(|\psi_1\rangle,|\psi_2\rangle,\cdots,|\psi_d\rangle)$

$R\cdots$ unitary　　$R^\dagger=R^{-1}$

$(R\psi)^\dagger(R\psi)=(\psi D)^\dagger(\psi D)=D^\dagger\psi^\dagger\psi D=D^\dagger D$

$=\psi^\dagger R^\dagger R\psi=\psi^\dagger\psi=E_d$

$\longrightarrow D^\dagger=D^{-1}$ unitary

$R_3=R_2R_1$　回転操作が群をつくる

$R_1\psi=\psi D(R_1)$　　$R_2R_1\psi=R_2\psi D(R_1)=\psi D(R_2)D(R_1)$

$R_3\psi=\psi D(R_3)=\psi D(R_2)D(R_1)\longrightarrow D(R_3)=D(R_2)D(R_1)$

何もしない操作　$D(1)=E_d$

$D(R^{-1})D(R)=D(1)=E_d$　　$D(R^{-1})=(D(R))^{-1}$

rotation — R — 抽象的

$d\times d$ matrix — $D(R)$ — 具体的　回転群の行列表現

無限小回転　$\delta\omega$方向, $|\delta\omega|$回転

$R=e^{-i\delta\omega\cdot L}=e^{-i\theta n\cdot J}\sim 1-i\theta n\cdot J$　　$\hbar\equiv1$

$D(R)=D(1-i\theta n\cdot J)$　行列表現

DはJの行列要素で決まる

$|\psi_1\rangle,|\psi_2\rangle,\cdots,|\psi_d\rangle$

J^2, J_zの同時固有状態　$d=2j+1$

$$\langle \psi_m | J_z | \psi_{m'} \rangle = m\,\delta_{mm'}$$
$$J_z |\psi_m\rangle = m |\psi_m\rangle$$
$$J_\pm |\psi_m\rangle = \sqrt{(j\mp m)(j\pm m+1)}\, |\psi_{m\pm1}\rangle$$
$$\langle \psi_{m'} | J_\pm | \psi_m \rangle = \sqrt{(j\mp m)(j\pm m+1)}\, \delta_{m',m\pm1}$$

一般の $D(R)$

$$R = R_\alpha(\mathbb{e}_z) R_\beta(\mathbb{e}_y) R_\gamma(\mathbb{e}_z) \qquad R_\theta(\mathbb{m}) = e^{-i\mathbb{m}\cdot\mathbb{J}}$$
$$D(R) = D(R_\alpha(\mathbb{e}_z))\, D(R_\beta(\mathbb{e}_y))\, D(R_\gamma(\mathbb{e}_z))$$
$$= D(e^{-i\alpha J_z})\, D(e^{-i\beta J_y})\, D(e^{-i\gamma J_z})$$

$$D(e^{-i\alpha J_z}) = \begin{pmatrix} e^{-i(-j)\alpha} & & & \\ & e^{-i(-j+1)\alpha} & & \\ & & \ddots & \\ & & & e^{-ij\alpha} \end{pmatrix}$$

$$[D(R)]_{m',m} = \langle \psi_{m'} | e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z} | \psi_m \rangle$$
$$= e^{-i\alpha m'} [e^{-i\beta J_y}]_{m',m}\, e^{-i\gamma m}$$

Schwinger Boson

$a_+,\ a_-$ $\qquad [a_+, a_+^+] = [a_-, a_-^+] = 1 \qquad [a_+, a_-] = 0$

$$a = \begin{pmatrix} a_+ \\ a_- \end{pmatrix} \qquad a^+ a = (a_+^+, a_-^+) \begin{pmatrix} a_+ \\ a_- \end{pmatrix} = a_+^+ a_+ + a_-^+ a_- = n_+ + n_-$$

$n_\pm = a_\pm^+ a_\pm$ number operator

$$\mathbb{J} = \frac{1}{2} a^+ \boldsymbol{\sigma} a$$

$$J_i = \frac{1}{2} a^+ \sigma_i a = \frac{1}{2} (a_+, a_-) \sigma_i \begin{pmatrix} a_+ \\ a_- \end{pmatrix} = \frac{1}{2} a_\alpha^+ (\sigma_i)_{\alpha\alpha'} a_{\alpha'}$$

$\mathbb{J} \to$ angular momentum $\quad [J_i, J_j] = i\varepsilon_{ijk} J_k$ を確認

$$[a_\alpha, a_\beta^+] = \delta_{\alpha\beta}$$

$$[J_i, J_j] = \frac{1}{4}[a_\alpha^+(\sigma_i)_{\alpha\beta}a_\beta, a_\gamma^+(\sigma_j)_{\gamma\delta}a_\delta]$$

$$= \frac{1}{4}a_\alpha^+(\sigma_i)_{\alpha\beta}[a_\beta, a_\gamma^+](\sigma_j)_{\gamma\delta}a_\delta$$

$$+ \frac{1}{4}a_\gamma^+(\sigma_j)_{\gamma\delta}[a_\alpha^+, a_\delta](\sigma_i)_{\alpha\beta}a_\beta$$

$$= \frac{1}{4}a_\alpha^+(\sigma_i)_{\alpha\beta}(\sigma_j)_{\beta\delta}a_\delta - \frac{1}{4}a_\gamma^+(\sigma_j)_{\gamma\alpha}(\sigma_i)_{\alpha\beta}a_\beta$$

$$= \frac{1}{4}a_\alpha^+(\sigma_i\sigma_j)_{\alpha\delta}a_\delta - \frac{1}{4}a_\gamma^+(\sigma_j\sigma_i)_{\gamma\beta}a_\beta$$

$$= \frac{1}{4}a^+(\sigma_i\sigma_j - \sigma_j\sigma_i)a$$

$i = j \rightarrow [J_i, J_j] = 0$

$i \neq j \rightarrow \sigma_j\sigma_i = \sigma_i\sigma_j$ $\qquad \sigma_i\sigma_j = i\varepsilon_{ijk}\sigma_k$

$i \neq j$の場合

$$[J_i, J_j] = \frac{1}{4}a^+(2\sigma_i\sigma_j)a = \frac{1}{2}a^+\sigma_i\sigma_j a$$

$$= \frac{1}{2}a^+ i\varepsilon_{ijk}\sigma_k a = i\varepsilon_{ijk}\frac{1}{2}a^+\sigma_k a = i\varepsilon_{ijk}J_k$$

$$J^2 = \frac{1}{2}(J_+J_- + J_-J_+) + J_z^2$$

$$J_z = a^+\frac{1}{2}\sigma_z a = \frac{1}{2}(a_+^+, a_-^+)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}a_+\\ a_-\end{pmatrix}$$

$$= \frac{1}{2}(a_+^+a_+ - a_-^+a_-) = \frac{1}{2}(n_+ - n_-)$$

$$J_\pm = J_x \pm iJ_y = a^+\frac{1}{2}(\sigma_x \pm i\sigma_y)a = a^+\frac{1}{2}\left\{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \pm \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right\}a$$

$$J_+ = a^+\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}a = a_+^+a_- \qquad (-) \rightarrow (+)$$

$$J_- = a^+\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}a = a_-^+a_+ \qquad (+) \rightarrow (-)$$

$$J^2=\frac{1}{2}(a_+^+a_-a_-^+a_+ + a_-^+a_+a_+^+a_-)+\frac{1}{4}(n_+-n_-)^2$$

$$=\frac{1}{2}\{n_+(1-n_-)+n_-(1+n_+)\}+\frac{1}{4}(n_++n_-)^2-n_+n_-$$

$$=\frac{1}{4}\{2n_+n_-+2n_++2n_-n_++2n_-+(n_++n_-)^2-4n_+n_-\}$$

$$=\frac{1}{4}\{2(n_++n_-)+(n_++n_-)^2\}=\frac{1}{4}(2n+n^2)=\frac{1}{4}n(n+2)$$

$$J^2=\frac{1}{2}n\left(\frac{1}{2}n+1\right)$$

$$=\hat{j}(\hat{j}+1)$$

$\hat{j}=\frac{1}{2}\hat{n}$ 全Boson数の半分

$$\hat{j}=\frac{1}{2}(\hat{n}_++\hat{n}_-)$$

$$\hat{J}_z=\frac{1}{2}(\hat{n}_+-\hat{n}_-)$$

$$|n_+,n_-\rangle=\frac{1}{\sqrt{n_+!}}\frac{1}{\sqrt{n_-!}}(a_+^+)^{n_+}(a_-^+)^{n_-}|0\rangle$$

$$\hat{n}_\pm|n_+,n_-\rangle=n_\pm|n_+,n_-\rangle$$

$$|j,m\rangle\equiv|n_+,n_-\rangle=\frac{1}{\sqrt{(j+m)!}}\frac{1}{\sqrt{(j-m)!}}(a_+^+)^{j+m}(a_-^+)^{j-m}|0\rangle$$

explicit form

$$j=\frac{1}{2}(n_++n_-)\qquad m=\frac{1}{2}(n_+-n_-)$$

$$n_+=j+m\qquad n_-=j-m$$