量子力学3まとめレポート　7/17分.

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3D rotation　　3次元回転.

$R(\alpha,\beta,\gamma) = R_\alpha(\hat{z}) R_\beta(\hat{y}) R_\gamma(\hat{z})$　　$\alpha.\beta.\gamma$　Euler angles　オイラー角.

$R_\theta(\hat{n})$: $\hat{n}$軸まわりの回転

・ψ : wave function

$R : \psi \mapsto R\psi$, $[H, R] = 0$　とする.

$\psi_1, \cdots, \psi_d$: d fold degenerated / complete　: d重に縮退した全ての状態.

$\rightarrow H|\psi_i\rangle = |\psi_i\rangle E$, $i = 1 \sim d$

$RH|\psi_i\rangle = (R|\psi_i\rangle)E = H(R|\psi_i\rangle)$

$R|\psi_i\rangle$: eigen state with energy E.

線型代数の原理から

$R|\psi_i\rangle = |\psi_j\rangle \underline{D_{ji}}$ 係数 coef.　: linear combination　線型結合

$= |\psi_i\rangle D_{ji}(R)$

$R(|\psi_1\rangle, |\psi_2\rangle, \cdots |\psi_d\rangle) = (|\psi_1\rangle, |\psi_2\rangle, \cdots |\psi_d\rangle)\, \mathbb{D}(R)$　d×d 行列"

$\psi \equiv (|\psi_1\rangle, \cdots, |\psi_d\rangle) \longrightarrow R\psi = \psi \mathbb{D}(R)$　と書ける.

R : unitary $\longleftrightarrow R^\dagger = R^{-1}$, $R^\dagger R = 1$　となっているならば,

$(R\psi)^\dagger (R\psi) = (\psi\mathbb{D})^\dagger(\psi\mathbb{D}) = \mathbb{D}^\dagger \psi^\dagger \psi \mathbb{D} = \mathbb{D}^\dagger \mathbb{D}$

$= \psi^\dagger R^\dagger R \psi$

$= \psi^\dagger \psi = \begin{pmatrix} \langle\psi_1| \\ \vdots \\ \langle\psi_d| \end{pmatrix} (|\psi_1\rangle, \cdots |\psi_d\rangle) = E_d$　: d次元単位行列.

$\rightarrow \mathbb{D}^\dagger = \mathbb{D}^{-1}$

回転操作が群をつくる.

R_1, R_2, $R_3 = R_2 R_1$

$R_1\psi = \psi \mathbb{D}(R_1)$

$R_2 R_1 \psi = R_2 \psi \mathbb{D}(R_1) = \psi \mathbb{D}(R_2)\mathbb{D}(R_1)$　$= R_3\psi = \psi\mathbb{D}(R_3)$

$\rightarrow \psi\mathbb{D}(R_3) = \psi\mathbb{D}(R_2)\mathbb{D}(R_1)$

$\mathcal{U}^\dagger$を作用させる ($\mathcal{U}^\dagger\mathcal{U} = Ed$) と, $\mathbb{D}(R_3) = \mathbb{D}(R_2)\mathbb{D}(R_1)$

何もしない回転 $\mathbb{D}(1) = Ed$, $\mathbb{D}(R^{-1})\mathbb{D}(R) = \mathbb{D}(1) = Ed \longrightarrow \mathbb{D}(R^{-1}) = (\mathbb{D}(R))^{-1}$

R	$\longleftrightarrow$	$\mathbb{D}(R)$
rotation		$d\times d$行列
abstract		concrete
抽象的		具体的

$\mathbb{D}(R)$: 回転群のd次元ユニタリ(行列)表現

$\mathbb{D}^\dagger = \mathbb{D}^{-1}$ representation

無限小回転 infinitesimal rotation

$(R = e^{-i\delta\vec{\omega}\cdot\vec{J}/\hbar}) \longrightarrow R = e^{-i\theta\hat{n}\cdot\vec{J}} \sim 1 - i\theta\hat{n}\cdot\vec{J}$ $\hbar = 1$

$\delta\vec{\omega}$方向回転軸
$|\delta\omega|$ 回転

$\hat{n}$方向 θ回転 (θ:無限小)

$\longrightarrow \mathbb{D}(R) = \mathbb{D}(1 - i\theta\hat{n}\cdot\vec{J})$ $\mathbb{D}$は$\vec{J}$の行列要素で決まる.

$|\psi_1\rangle, \cdots |\psi_d\rangle$, $d = 2j+1$ J^2, J_zの同時固有状態を採用 $\longrightarrow$ spin j表現.

$$\left.\begin{aligned}
&\langle\psi_m|J_z|\psi_{m'}\rangle = m\,\delta_{mm'} \\
&\longleftrightarrow J_z|\psi_m\rangle = m|\psi_m\rangle \\
&\langle\psi_{m'}|J_\pm|\psi_m\rangle = \sqrt{(j\mp m)(j\pm m+1)}\,\delta_{m',m\pm1}
\end{aligned}\right) \longrightarrow$$

このとき一般の$\mathbb{D}(R)$はどのように表わされるか?

$R = R_\alpha(\hat{z})R_\beta(\hat{y})R_\gamma(\hat{z})$

$$\mathbb{D}(R) = \mathbb{D}(R_\alpha(\hat{z}))\,\mathbb{D}(R_\beta(\hat{y}))\,\mathbb{D}(R_\gamma(\hat{z}))$$
$$= \mathbb{D}(e^{-i\alpha J_z})\,\mathbb{D}(e^{-i\beta J_y})\,\mathbb{D}(e^{-i\gamma J_z}) \qquad (R_\theta(\hat{n}) = e^{-i\hat{n}\cdot\vec{J}})$$

$$\mathbb{D}(e^{-i\alpha J_z}) = \begin{pmatrix} e^{-i(-j)\alpha} & & & \\ & e^{-i(-j+1)\alpha} & & \\ & & \ddots & \\ & & & e^{-ij\alpha}\end{pmatrix} = diag(e^{ij\alpha}, \cdots, e^{-ij\alpha})$$

$$\underset{R_\alpha(\hat{z})R_\beta(\hat{y})R_\gamma(\hat{z})}{[\mathbb{D}(R)]_{m'm}} = \langle\psi_{m'}|e^{-i\alpha J_z}e^{-i\beta J_y}e^{-i\gamma J_z}|\psi_m\rangle = e^{-i\alpha m'}\underbrace{[e^{-i\beta J_y}]_{m'm}}_{d_{m'm}(\beta)}e^{-i\gamma m}$$

これを求めることが目標.

☆ $d^j_{m'm}(\beta)$

Schwinger − Boson による方法

a_+, a_- : Bosons , $\left(\begin{array}{l} [a_+, a_+^\dagger] = 1 , [a_-, a_-^\dagger] = 1 \\ [a_+, a_-] = 0 \end{array} \right.$

$a = \begin{pmatrix} a_+ \\ a_- \end{pmatrix}$, $a^\dagger a = (a_+^\dagger, a_-^\dagger)\begin{pmatrix} a_+ \\ a_- \end{pmatrix} = a_+^\dagger a_+ + a_-^\dagger a_- = n_+ + n_-$

$n_\pm = a_\pm^\dagger a_\pm$: ± Boson の number operation

$\vec{J} \equiv \frac{1}{2} a^\dagger \vec{\sigma} a$ ここで $[a_\alpha, a_\beta^\dagger] = \delta_{\alpha\beta}$ であるから.

$J_i = \frac{1}{2} a^\dagger \sigma_i a = \frac{1}{2}(a_+, a_-)\sigma_i \begin{pmatrix} a_+ \\ a_- \end{pmatrix} = \frac{1}{2} a_\alpha^\dagger (\sigma_j)_{\alpha\alpha'} a_{\alpha'}$

元々, $\vec{J}$: angular momentum であることから.

交換関係 $[J_i, J_j] = i \varepsilon_{ijk} J_k$ を確認する必要がある。

$$
\begin{aligned}
[J_i, J_j] &= \frac{1}{4}[a_\alpha^\dagger (\sigma_i)_{\alpha\beta} a_\beta , a_\gamma^\dagger (\sigma_j)_{\gamma\delta} a_\delta] \\
&= \frac{1}{4} a_\alpha^\dagger (\sigma_i)_{\alpha\beta} \underbrace{[a_\beta, a_\gamma^\dagger]}_{\delta_{\beta\gamma}} (\sigma_j)_{\gamma\delta} a_\delta + \frac{1}{4} a_\gamma^\dagger (\sigma_j)_{\gamma\delta} \underbrace{[a_\alpha^\dagger, a_\delta]}_{-\delta_{\alpha\delta}} (\sigma_i)_{\alpha\beta} a_\beta \\
&\quad (\because [AB, C] = A[B.C] + [A.C]B) \\
&= \frac{1}{4} a_\alpha^\dagger (\sigma_i)_{\alpha\beta} (\sigma_j)_{\beta\delta} a_\delta - \frac{1}{4} a_\gamma^\dagger (\sigma_j)_{\gamma\alpha} (\sigma_i)_{\alpha\beta} a_\beta \\
&= \frac{1}{4} a_\alpha^\dagger (\sigma_i \sigma_j)_{\alpha\delta} a_\delta - \frac{1}{4} a_\gamma^\dagger (\sigma_j \sigma_i)_{\gamma\beta} a_\beta \\
&= \frac{1}{4} a^\dagger (\sigma_i \sigma_j - \sigma_j \sigma_i) a
\end{aligned}
$$

$i = j$ $[J_i, J_j] = 0$

$i \neq j$ $[J_i, J_j] = \frac{1}{\cancel{4}2} a^\dagger \cancel{2} \sigma_i \sigma_j a = \frac{1}{2} a^\dagger i \varepsilon_{ijk} \sigma_k a = i \varepsilon_{ijk} \frac{1}{2} \underbrace{a^\dagger \sigma_k a}_{J_k} = i\varepsilon_{ijk} J_k$

$$J^2 = \vec{J}\cdot\vec{J} = \frac{1}{2}(J_+J_- + J_-J_+) + J_z^2$$

$$J_z^2 = a^\dagger \frac{1}{2}\sigma_z a = \frac{1}{2}(a_+^\dagger, a_-^\dagger)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}a_+\\ a_-\end{pmatrix}$$

$$= \frac{1}{2}(a_+^\dagger a_+ - a_-^\dagger a_-)$$

$$= \frac{1}{2}(n_+ - n_-)$$

$$J_\pm = J_x \pm iJ_y = a^\dagger \frac{1}{2}(\sigma_x \pm i\sigma_y) a = a^\dagger \frac{1}{2}\left\{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \pm \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right\} a$$

$$\begin{cases} J_+ = a^\dagger\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} a = a_+^\dagger a_- \\ J_- = a^\dagger\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} a = a_-^\dagger a_+ \end{cases}$$

$$J^2 = \frac{1}{2}(\underbrace{a_+^\dagger a_- a_-^\dagger a_+}_{= a_+^\dagger a_+ a_- a_-^\dagger = n_+(1+n_-)} + \underbrace{a_-^\dagger a_+ a_+^\dagger a_-}_{= a_-^\dagger a_- a_+ a_+^\dagger = n_-(1+n_+)}) + \frac{1}{4}(n_+ - n_-)^2$$

$$= \frac{1}{2}\{n_+(n_-+1) + n_-(n_++1)\} + \frac{1}{4}(n_++n_-)^2 - n_+n_-$$

$$= \frac{1}{4}(2n_+n_- + 2n_+ + 2n_-n_+ + 2n_- + (n_++n_-)^2 - 4n_+n_-)$$

$$= \frac{1}{4}(2(n_++n_-) + (n_++n_-)^2) \quad = \frac{1}{4}(2n+n^2) \quad = \frac{1}{4}n(n+2)$$

$J^2 = \frac{1}{2}n(\frac{1}{2}n+1)$ 　　$\hat{j} = \frac{1}{2}\hat{n}$ としておくと，（$\hat{j}$：全Boson数の半分）

$$J^2 = \hat{j}(\hat{j}+1)$$

$$\hat{j} = \frac{1}{2}(\hat{n}_+ + \hat{n}_-)$$

ここで，$\hat{J}_z = \frac{1}{2}(\hat{n}_+ - \hat{n}_-)$ を使うと，

$$|n_+, n_-\rangle = \frac{1}{\sqrt{n_+!}}\frac{1}{\sqrt{n_-!}}(a_+^\dagger)^{n_+}(a_-)^{n_-}|0\rangle$$

$$\hat{n}_\pm |n_+, n_-\rangle = n_\pm |n_+, n_-\rangle$$

$$|j, m\rangle \equiv |n_+, n_-\rangle = \frac{1}{\sqrt{(j+m)!}}\frac{1}{\sqrt{(j-m)!}}(a_+^\dagger)^{j+m}(a_-)^{j-m}|0\rangle$$: explicit form

$$\left.\begin{aligned} j &= \frac{1}{2}(n_+ + n_-) \\ m &= \frac{1}{2}(n_+ - n_-) \end{aligned}\right\} \longrightarrow \begin{aligned} n_+ &= j+m \\ n_- &= j-m \end{aligned}$$