量子力学3（演習問題3）

201110844
荻野孝浩

問題1

(1) $p_i = -i\hbar\partial_i$

$$[r_i, p_j] = r_i p_j - p_j r_i$$
$$= -i\hbar(r_i\partial_j - \partial_j r_i)$$
$$= -i\hbar(r_i\partial_j - \delta_{ij} - r_i\partial_j)$$
$$= i\hbar\delta_{ij}$$

(2)
$$[L_i, L_j] = [\varepsilon_{iab} r_a p_b, \varepsilon_{jcd} r_c p_d]$$
$$= \varepsilon_{iab}\varepsilon_{jcd}[r_a p_b, r_c p_d]$$
$$= \varepsilon_{iab}\varepsilon_{jcd}(r_a[p_b, r_c p_d] + [r_a, r_c p_d]p_b)$$
$$= \varepsilon_{iab}\varepsilon_{jcd}(r_a r_c[p_b, p_d] + r_a[p_b, r_c]p_d + r_c[r_a, p_d]p_b + [r_a, r_c]p_d p_b)$$
$$= \varepsilon_{iab}\varepsilon_{jcd}\{r_a(-i\hbar\delta_{bc})p_d + r_c(i\hbar\delta_{ad})p_b\}$$
$$= i\hbar(-\varepsilon_{iab}\varepsilon_{jbd} r_a p_d + \varepsilon_{idb}\varepsilon_{jcd} r_c p_b)$$
$$= i\hbar(\varepsilon_{iab}\varepsilon_{jdb} r_a p_d - \varepsilon_{ibd}\varepsilon_{jcd} r_c p_b)$$
$$= i\hbar\{(\delta_{ij}\delta_{ad} - \delta_{id}\delta_{aj}) r_a p_d - (\delta_{ij}\delta_{bc} - \delta_{ic}\delta_{bj}) r_c p_b\}$$
$$= i\hbar\{\delta_{ij}(\vec{r}\cdot\vec{p}) - r_j p_i - \delta_{ij}(\vec{r}\cdot\vec{p}) + r_i p_j\}$$
$$= i\hbar(r_i p_j - r_j p_i)$$
$$= i\hbar\varepsilon_{ijk} L_k$$

(3)
$$[L_i, r_j] = [\varepsilon_{iab} r_a p_b, r_j]$$
$$= \varepsilon_{iab}(r_a[p_b, r_j] + [r_a, r_j]p_b)$$
$$= \varepsilon_{iab} r_a(-i\hbar\delta_{jb})$$
$$= -i\hbar\varepsilon_{iaj} r_a$$
$$= i\hbar\varepsilon_{ija} r_a$$

(4)
$$[L_i, p_j] = [\varepsilon_{iab} r_a p_b, p_j]$$
$$= \varepsilon_{iab}(r_a[p_b, p_j] + [r_a, p_j]p_b)$$
$$= \varepsilon_{iab}(i\hbar\delta_{aj})p_b$$
$$= i\hbar\varepsilon_{ijb} p_b$$

問題2

(1) $[J^2, J_z] = [J_x^2 + J_y^2 + J_z^2, J_z]$
$= [J_x^2, J_z] + [J_y^2, J_z] + [J_z^2, J_z]$
$= J_x[J_x, J_z] + [J_x, J_z]J_x$
$+ J_y[J_y, J_z] + [J_y, J_z]J_y$
$+ J_z[J_z, J_z] + [J_z, J_z]J_z$
$= J_x(-i\hbar J_y) - i\hbar J_y J_x + J_y(i\hbar J_x) + i\hbar J_x J_y$
$= i\hbar(J_x J_y - J_y J_x + J_y J_x + J_x J_y) = 0$

(2)
$J_\pm = J_x \pm i J_y$

$[J_z, J_\pm] = [J_z, J_x] \pm i[J_z, J_y]$
$= i\hbar J_y \mp i(i\hbar J_x)$
$= \pm\hbar(J_x \pm iJ_y) = \pm\hbar J_\pm$

(3)
$[J_+, J_-] = [J_x + iJ_y, J_x - iJ_y]$
$= [J_x, -iJ_y] + [iJ_y, J_x]$
$= -i(i\hbar J_z) + i(-i\hbar J_z) = 2\hbar J_z$

(4)
$\varepsilon_{ijk}[J_j, J_k] = \varepsilon_{ijk} J_j J_k - \varepsilon_{ijk} J_k J_j$
$= \varepsilon_{ijk} J_j J_k + \varepsilon_{ijk} J_j J_k$
$= 2\varepsilon_{ijk} J_j J_k$
$= 2(\mathbb{J} \times \mathbb{J})_i$

また
$[J_j, J_k] = i\hbar\varepsilon_{jka} J_a$ より

$\varepsilon_{ijk}[J_j, J_k] = i\hbar\varepsilon_{ijk}\varepsilon_{jka} J_a$
$= i\hbar\varepsilon_{jki}\varepsilon_{jka} J_a$
$= i\hbar(2\delta_{ia}) J_a$
$= 2i\hbar J_i$

∴

$2(\mathbb{J} \times \mathbb{J})_i = 2i\hbar J_i$
$\Rightarrow (\mathbb{J} \times \mathbb{J})_i = i\hbar J_i$

よって $\mathbb{J} \times \mathbb{J} = i\hbar\mathbb{J}$

Date No.

③

(5)

$$[\boldsymbol{J}_1\cdot\boldsymbol{J}_2,\ J_{1z}+J_{2z}]$$
$$= [J_{1x}J_{2x}+J_{1y}J_{2y}+J_{1z}J_{2z},\ J_{1z}+J_{2z}]$$
$$= J_{1x}[J_{2x}, J_{1z}+J_{2z}] + [J_{1x}, J_{1z}+J_{2z}]J_{2x}$$
$$+ J_{1y}[J_{2y}, J_{1z}+J_{2z}] + [J_{1y}, J_{1z}+J_{2z}]J_{2y}$$
$$+ J_{1z}[J_{2z}, J_{1z}+J_{2z}] + [J_{1z}, J_{1z}+J_{2z}]J_{2z}$$
$$= J_{1x}[J_{2x}, J_{2z}] + [J_{1x}, J_{1z}]J_{2x} + J_{1y}[J_{2y}, J_{2z}] + [J_{1y}, J_{1z}]J_{2y}$$
$$+ J_{1z}[J_{2z}, J_{2z}] + [J_{1z}, J_{1z}]J_{2z}$$
$$= J_{1x}(-i\hbar J_{2y}) - i\hbar J_{1y}J_{2x} + J_{1y}(i\hbar J_{2x}) + i\hbar J_{1x}J_{2y}$$
$$= i\hbar(-J_{1x}J_{2y} - J_{1y}J_{2x} + J_{1y}J_{2x} + J_{1x}J_{2y})$$
$$= 0$$

(6)

$$[\boldsymbol{J}^2, J_z] = [(\boldsymbol{J}_1+\boldsymbol{J}_2)^2, J_z] \qquad (\boldsymbol{J}_1, \boldsymbol{J}_2 \text{ は可換})$$
$$= [\boldsymbol{J}_1^2+\boldsymbol{J}_2^2+2\boldsymbol{J}_1\cdot\boldsymbol{J}_2, J_z]$$
$$= [\boldsymbol{J}_1^2+\boldsymbol{J}_2^2, J_z] + 2[\boldsymbol{J}_1\cdot\boldsymbol{J}_2, J_{1z}+J_{2z}]$$
$$= [\boldsymbol{J}_1^2, J_z] + [\boldsymbol{J}_2^2, J_z]$$
$$= [\boldsymbol{J}_1^2, J_{1z}] + [\boldsymbol{J}_1^2, J_{2z}] + [\boldsymbol{J}_2^2, J_{1z}] + [\boldsymbol{J}_2^2, J_{2z}]$$
$$= [\boldsymbol{J}_1^2, J_{1z}] + [\boldsymbol{J}_2^2, J_{2z}]$$
$$= 0 + 0 \qquad (\because (1))$$
$$= 0$$

Date　　　　　　　　No.

問題3

(1)

$$\mathbb{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos\phi\sin\theta \\ r\sin\phi\sin\theta \\ r\cos\theta \end{pmatrix}$$

$$\partial_r \mathbb{r} = \begin{pmatrix} \cos\phi\sin\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{pmatrix} \Rightarrow \mathbb{e}_r = \partial_r \mathbb{r}$$

$$\partial_\theta \mathbb{r} = \begin{pmatrix} r\cos\phi\cos\theta \\ r\sin\phi\cos\theta \\ -r\sin\theta \end{pmatrix} \Rightarrow \mathbb{e}_\theta = \frac{1}{r}\partial_\theta \mathbb{r}$$

$$\partial_\phi \mathbb{r} = \begin{pmatrix} -r\sin\phi\sin\theta \\ r\cos\phi\sin\theta \\ 0 \end{pmatrix} \Rightarrow \mathbb{e}_\phi = \frac{1}{r\sin\theta}\partial_\phi \mathbb{r}$$

$$\therefore (\mathbb{e}_r, \mathbb{e}_\theta, \mathbb{e}_\phi) = (\widehat{\partial_r \mathbb{r}}, \widehat{\partial_\theta \mathbb{r}}, \widehat{\partial_\phi \mathbb{r}})$$
$$= \begin{pmatrix} \cos\phi\sin\theta & \cos\phi\cos\theta & -\sin\phi \\ \sin\phi\sin\theta & \sin\phi\cos\theta & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}$$

(2)

$$\mathbb{L} = \mathbb{r} \times \mathbb{p}$$
$$= r\mathbb{e}_r \times (-i\hbar\nabla)$$
$$= -i\hbar r\, \mathbb{e}_r \times \left(\mathbb{e}_r \partial_r + \mathbb{e}_\theta \frac{1}{r}\partial_\theta + \mathbb{e}_\phi \frac{1}{r\sin\theta}\partial_\phi\right)$$
$$= -i\hbar r\left(\mathbb{e}_\phi \frac{1}{r}\partial_\theta - \mathbb{e}_\theta \frac{1}{r\sin\theta}\partial_\phi\right)$$
$$= i\hbar\left(\mathbb{e}_\theta \frac{1}{\sin\theta}\partial_\phi - \mathbb{e}_\phi \partial_\theta\right)$$

(3)

$$
\begin{aligned}
L_+ &= L_x + i L_y \\
&= i\hbar\left(\cos\phi\cos\theta\,\frac{1}{\sin\theta}\,\partial_\phi + \sin\phi\,\partial_\theta\right) - \hbar\left(\sin\phi\cos\theta\,\frac{1}{\sin\theta}\,\partial_\phi - \cos\phi\,\partial_\theta\right) \\
&= \hbar\left(i\cos\phi\cot\theta\,\partial_\phi + i\sin\phi\,\partial_\theta - \sin\phi\cot\theta\,\partial_\phi + \cos\phi\,\partial_\theta\right) \\
&= \hbar\left(\cos\phi + i\sin\phi\right)\left(\partial_\theta + i\cot\theta\,\partial_\phi\right) \\
&= \hbar e^{i\phi}\left(\partial_\theta + i\cot\theta\,\partial_\phi\right)
\end{aligned}
$$

$$
\begin{aligned}
L_- &= L_x - i L_y \\
&= \hbar\left(i\cos\phi\cot\theta\,\partial_\phi + i\sin\phi\,\partial_\theta + \sin\phi\cot\theta\,\partial_\phi - \cos\phi\,\partial_\theta\right) \\
&= \hbar\left(\cos\phi - i\sin\phi\right)\left(-\partial_\theta + i\cot\theta\,\partial_\phi\right) \\
&= \hbar e^{-i\phi}\left(-\partial_\theta + i\cot\theta\,\partial_\phi\right)
\end{aligned}
$$

(4)

$$L_z Y_{\ell m} = \hbar m Y_{\ell m}$$

$$\Rightarrow i\hbar\left(-\sin\theta\,\frac{1}{\sin\theta}\,\partial_\phi\right)\Theta_{\ell m}(\theta)\Phi_m(\phi) = \hbar m\,\Theta_{\ell m}(\theta)\Phi_m(\phi)$$

$$\Rightarrow \partial_\phi \Phi_m(\phi) = i m \Phi_m(\phi)$$

$$\Rightarrow \Phi_m(\phi) = A e^{im\phi} \qquad (A：定数)$$

$$\int_0^{2\pi} d\phi\,|\Phi_m(\phi)|^2 = |A|^2\int_0^{2\pi} d\phi\; e^{-im\phi}e^{im\phi} = 2\pi|A|^2 = 1$$

$$\Rightarrow A = \frac{1}{\sqrt{2\pi}}$$

$\therefore$

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}$$

(5)

$$L_+ Y_{\ell\ell} = \hbar e^{i\phi}\left(\partial_\theta + i\cot\theta\,\partial_\phi\right)\Theta_{\ell\ell}(\theta)\Phi_\ell(\phi) = 0$$

$$\Rightarrow \Phi_\ell(\phi)\,\partial_\theta\Theta_{\ell\ell}(\theta) + i\cot\theta\,\Theta_{\ell\ell}(\theta)\,\partial_\phi\Phi_\ell(\phi) = 0$$

$$\Rightarrow i\,\frac{\partial_\phi\Phi_\ell(\phi)}{\Phi_\ell(\phi)} = -\frac{\partial_\theta\Theta_{\ell\ell}(\theta)}{\cot\theta\;\Theta_{\ell\ell}(\theta)}$$

$$\Rightarrow i(i\ell) = -\frac{\partial_\theta\Theta_{\ell\ell}(\theta)}{\cot\theta\;\Theta_{\ell\ell}(\theta)}$$

$$\Rightarrow \partial_\theta\Theta_{\ell\ell}(\theta) - \ell\cot\theta\;\Theta_{\ell\ell}(\theta) = 0$$

(6) $\Theta_{\ell\ell} = C_\ell \sin^\ell\theta$ とおくと

$$\partial_\theta \Theta_{\ell\ell} = C_\ell\, \ell \sin^{\ell-1}\theta \cos\theta$$

$$\ell\cot\theta\, \Theta_{\ell\ell} = C_\ell\, \ell \frac{\cos\theta}{\sin\theta} \sin^\ell\theta = C_\ell\, \ell \sin^{\ell-1}\theta \cos\theta$$

$\therefore$
$$\partial_\theta \Theta_{\ell\ell} = \ell\cot\theta\, \Theta_{\ell\ell}$$

よって
$\partial_\theta \Theta_{\ell\ell} - \ell\cot\theta\, \Theta_{\ell\ell} = 0$ の解は
$$\Theta_{\ell\ell} = C_\ell \sin^\ell\theta$$

(7) $\int_0^\pi d\theta \sin\theta \left|\Theta_{\ell\ell}(\theta)\right|^2$

$$= \int_0^\pi d\theta \sin\theta\, |C_\ell|^2 \sin^{2\ell}\theta$$

$$= |C_\ell|^2 \int_0^\pi d\theta \sin^{2\ell+1}\theta \qquad (z = \cos\theta)$$

$$= |C_\ell|^2 \int_1^{-1} (-dz)(1-z^2)^\ell$$

$$= |C_\ell|^2 \int_{-1}^1 dz\, (1-z^2)^\ell$$

$$= |C_\ell|^2 \left\{ \left[z(1-z^2)^\ell \right]_{-1}^1 - \int_{-1}^1 dz\, z\,\ell\,(1-z^2)^{\ell-1}(-2z) \right\}$$

$$= |C_\ell|^2 \cdot 2\ell \int_{-1}^1 dz\, z^2 (1-z^2)^{\ell-1}$$

$$= |C_\ell|^2 \cdot 2\ell \left\{ -\int_{-1}^1 dz\, \frac{z^3}{3} (\ell-1)(1-z^2)^{\ell-2}(-2z) \right\}$$

$$= |C_\ell|^2 \frac{2^2 \ell(\ell-1)}{3} \int_{-1}^1 dz\, z^4 (1-z^2)^{\ell-2}$$

$$= |C_\ell|^2 \frac{2^3 \ell(\ell-1)(\ell-2)}{3\cdot 5} \int_{-1}^1 dz\, z^6 (1-z^2)^{\ell-3}$$

$$= |C_\ell|^2 \frac{2^\ell \ell!}{3\cdot 5\cdot 7 \cdots 2\ell-1} \int_{-1}^1 dz\, z^{2\ell}$$

$$= |C_\ell|^2 \frac{(2^\ell \ell!)^2}{(2\ell+1)!} \left[z^{2\ell+1} \right]_{-1}^1$$

$l = 0, 1, 2 \cdots$ で

$$\int_0^\pi d\theta \sin\theta \left|\Theta_{l l}(\theta)\right|^2 = |C_l|^2 \frac{(2^l l!)^2}{(2l+1)!} \cdot 2 = 1$$

$$\therefore \quad |C_l|^2 = \frac{(2l+1)!}{2} \cdot \frac{1}{(2^l l!)^2}$$

$$\Rightarrow \left|(-)^l C_l'\right|^2 = |C_l'|^2 = \frac{(2l+1)!}{2} \frac{1}{(2^l l!)^2} \qquad (C_l' > 0)$$

$$\Rightarrow C_l' = \sqrt{\frac{(2l+1)!}{2}} \cdot \frac{1}{2^l l!}$$

よって $\quad C_l = (-)^l \sqrt{\frac{(2l+1)!}{2}} \cdot \frac{1}{2^l l!}$

(8)

$$Y_{00} = (-)^{\frac{0}{2}} \sqrt{\frac{1}{2} \frac{0!}{0!}} \frac{e^0}{\sqrt{2\pi}} \sin^0\theta \, P_0(\cos\theta)$$

$$= \sqrt{\frac{1}{4\pi}} P_0(\cos\theta)$$

$$= \frac{1}{2\sqrt{\pi}}$$

$$Y_{10} = \sqrt{\frac{3}{2}} \cdot \frac{1}{\sqrt{2\pi}} P_1(\cos\theta)$$

$$= \frac{1}{2}\sqrt{\frac{3}{\pi}} \cos\theta$$

$$Y_{1\pm1} = (-)^{\frac{\pm1+1}{2}} \sqrt{\frac{3}{2} \frac{(1-1)!}{(1+1)!}} \frac{e^{\pm i\phi}}{\sqrt{2\pi}} \sin\theta \frac{d}{d\cos\theta} P_1(\cos\theta)$$

$$= \mp \frac{\sqrt{3}}{2} \frac{1}{\sqrt{2\pi}} \sin\theta \, e^{\pm i\phi}$$

$$= \mp \frac{1}{2} \sqrt{\frac{3}{2\pi}} \sin\theta \, e^{\pm i\phi}$$

Date No.

$$Y_{20} = \sqrt{\frac{5}{2}\frac{2!}{2!}}\frac{1}{\sqrt{2\pi}} P_2(\cos\theta)$$

$$= \frac{1}{2}\sqrt{\frac{5}{\pi}}\frac{1}{2}(3\cos^2\theta - 1)$$

$$= \frac{1}{4}\sqrt{\frac{5}{\pi}}(3\cos^2\theta - 1)$$

$$Y_{2,\pm1} = \mp\sqrt{\frac{5}{2}\cdot\frac{1}{3!}}\frac{e^{\pm i\phi}}{\sqrt{2\pi}}\sin\theta\frac{d}{d\cos\theta}\left(P_2(\cos\theta)\right)$$

$$= \mp\frac{1}{2}\sqrt{\frac{5}{6\pi}}\sin\theta\,\frac{1}{2}\cdot 6\cos\theta\; e^{\pm i\phi}$$

$$= \mp\frac{3}{2}\sqrt{\frac{5}{6\pi}}\sin\theta\cos\theta\; e^{\pm i\phi}$$

$$Y_{2,\pm2} = (-)^{\frac{\pm2+2}{2}}\sqrt{\frac{5}{2}\frac{0!}{4!}}\frac{e^{\pm 2i\phi}}{\sqrt{2\pi}}\sin^2\theta\left(\frac{d}{d\cos\theta}\right)^2 P_2(\cos\theta)$$

$$= \frac{1}{2}\sqrt{\frac{5}{24\pi}}\sin^2\theta\cdot 3\,e^{\pm 2i\phi}$$

$$= \frac{3}{4}\sqrt{\frac{5}{6\pi}}\sin^2\theta\; e^{\pm 2i\phi}$$