量子力学3 7/5

20111086
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群の表現

対称操作 R $\quad H \mapsto RHR^{-1} = H' = H$

$R^\dagger R = 1$ $\quad \Rightarrow RH = HR \quad [H, R] = 0$

R: unitary

$H|\psi_j\rangle = |\psi_j\rangle E$ $\qquad R|\psi_j\rangle$

縮退度 d $\quad j = 1, 2, \cdots, d$ $\qquad HR|\psi_j\rangle = R\underbrace{H|\psi_j\rangle}_{=|\psi_j\rangle E} = R|\psi_j\rangle E$

(縮退したすべてを含む)

$R|\psi_j\rangle$: $\{|\psi_1\rangle, \cdots, |\psi_d\rangle\}$ の線型結合で書ける

$R|\psi_j\rangle = \sum_i |\psi_i\rangle D_{ij}(R) = (\psi D)_j$

$D_{ij}(R)$ ← 線型結合の係数

$\{R|\psi_1\rangle, R|\psi_2\rangle, \cdots, R|\psi_d\rangle\}$

$= R\{|\psi_1\rangle, \cdots |\psi_d\rangle\}$ $\qquad \Leftarrow \psi = \{|\psi_1\rangle, \cdots |\psi_d\rangle\}$

$= R\psi$

$= \{(\psi D)_1, (\psi D)_2, \cdots (\psi D)_d\}$

$= \psi D(R)$

$R\psi = \psi D(R)$

$R\{|\psi_1\rangle, \cdots |\psi_d\rangle\} = \{|\psi_1\rangle, \cdots, |\psi_d\rangle\} D(R)$

対称操作の集合 $\{R\}$ は群をつくる $\qquad R_2 R_1 = {}^\exists R_{21}$

$R_1\psi = \psi D(R_1)$, $\quad R_{21}\psi = \psi D(R_{21})$

$\underbrace{R_2 R_1}_{R_{21}}\psi = \underbrace{R_2\psi}_{=\psi D(R_2)} D(R_1) = \psi D(R_2) D(R_1)$

$\begin{cases} R_{21}\psi = \psi D(R_{21}) \\ \quad = \psi D(R_2) D(R_1) \end{cases}$

$D(R_2) D(R_1) = D(R_{21})$

$D(e) = E_d$, $\quad D(R^{-1}) = [D(R)]^{-1}$

$D(R)$: 群 $\{R\}$ の行列によるユニタリ表現

$R\psi = \psi D \Longleftrightarrow \psi^\dagger R^\dagger = D^\dagger \psi^\dagger$

$\psi^\dagger R^\dagger R \psi = \psi^\dagger \psi = E_d$

$= D^\dagger \psi^\dagger \psi D$

$= D^\dagger D \longrightarrow D^\dagger D = E_d \quad \Rightarrow D^\dagger = D^{-1}$, D: unitary

$\psi = (|\psi_1\rangle, |\psi_2\rangle \cdots, |\psi_d\rangle)$ $\qquad \psi^\dagger\psi = E_d \quad (\psi^\dagger\psi)_{ij} = \langle\psi_i|\psi_j\rangle$

$$\psi^\dagger\psi = \begin{pmatrix} \langle\psi_1| \\ \vdots \\ \langle\psi_d| \end{pmatrix} (|\psi_1\rangle, \cdots, |\psi_d\rangle) = \begin{pmatrix} \langle\psi_1|\psi_1\rangle & \langle\psi_1|\psi_2\rangle & \cdots & \\ \vdots & & \ddots & \vdots \\ \langle\psi_d|\psi_1\rangle & \cdots & & \langle\psi_d|\psi_d\rangle \end{pmatrix}$$

☆回転群の表現 $(\hbar = 1)$ ← 自然単位

$R_\alpha(\hat{n}) = e^{-i\alpha\hat{n}\cdot\mathbb{J}}$, $[J_i, J_j] = i\varepsilon_{ijk}J_k$

$\hookrightarrow D(R)$: 行列

無限小変換 $\alpha = \delta\alpha$: 無限小、$e^{-i\delta\alpha\hat{n}\cdot\mathbb{J}} = 1 - i\delta\alpha\hat{n}\cdot\mathbb{J}$

$R\psi = \psi D$

$(1 - i\delta\alpha\hat{n}\cdot\mathbb{J})\psi = \psi D(1 - i\delta\alpha\hat{n}\cdot\mathbb{J})$

$\doteq 1 - i\delta\alpha D(\hat{n}\cdot\mathbb{J})$

$J_x, J_y, J_z \to J_\pm, J_z$ を考えれば十分

$J_z\psi = \psi D(J_z)$

$J_z\{|\psi_1\rangle, \cdots, |\psi_d\rangle\} = \{|\psi_1\rangle, \cdots, |\psi_d\rangle\} D(J_z)$

$$J_z\{|\psi_{-j}\rangle, |\psi_{-j+1}\rangle, \cdots, |\psi_m\rangle, \cdots, |\psi_j\rangle\} = (\cdots, |\psi_m\rangle, \cdots) \begin{pmatrix} -h_j & & 0 \\ & h_m & \\ 0 & & h_j \end{pmatrix}$$

$J_z|\psi_m\rangle = m|\psi_m\rangle$

$= |\psi_{m'}\rangle D_{m'm}(J_z)$, $D_{m'm}(J_z) = m\delta_{m'm}$ $\underline{*1}$

$J_\pm|\psi_m\rangle = \sqrt{(j\mp m)(j\pm m+1)}\,|\psi_{m\pm1}\rangle$

$= |\psi_{m'}\rangle D^j_{m'm}(J_\pm)$, $D^j_{m'm}(J_\pm) = \sqrt{(j\mp m)(j\pm m+1)}\,\delta_{m',m\pm1}$ $\underline{*2}$

$*1, *2$ で定まる行列

$D^j(J_z)$, $D^j(J_\pm)$ $2j+1$ 次元行列, spin-j 表現

一般の回転 Euler角 (α, β, γ) で定まる

$R(\alpha,\beta,\gamma) = R_\alpha(z)R_\beta(y)R_\gamma(z) = e^{-i\alpha J_z}e^{-i\beta J_y}e^{-i\gamma J_z}$

$D^j(R(\alpha,\beta,\gamma)) = D^j(e^{-i\alpha J_z}e^{-i\beta J_y}e^{-i\gamma J_z})$

$= D^j(e^{-i\alpha J_z})D^j(e^{-i\beta J_y})D^j(e^{-i\gamma J_z})$

$|\psi_m\rangle = |jm\rangle$ $m : -j \to j$

$$D^j_{m'm}(R) = \langle jm'|R|jm\rangle = \underbrace{\langle jm'|e^{-i\alpha J_z}}_{\doteq e^{-i\alpha m'}\langle jm'|} e^{-i\beta J_y} \underbrace{e^{-i\gamma J_z}|jm\rangle}_{\doteq |jm\rangle e^{-i\gamma m}}$$

$$= e^{-i\alpha m'}\langle jm'|e^{-i\beta J_y}|jm\rangle e^{-i\gamma m}$$

$$D^j_{m'm}(R(\alpha,\beta,\gamma)) = e^{-i\alpha m'} d^j_{m'm} e^{-i\gamma m}$$

$$\underset{\uparrow}{d^j_{m'm}} = \langle jm'|e^{-i\beta J_y}|jm\rangle$$

Schwinger Bosonを用いた$d^j_{m'm}$の決定

Boson　　　　$a^\dagger a = \hat{n}$: 粒子数演算子

$$[a, a^\dagger] = 1,\ [a, a] = 0,\ [a^\dagger, a^\dagger] = 0$$

$$a|0\rangle = 0 \quad |n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle \rightarrow a^\dagger a|n\rangle = n|n\rangle$$

+, − の2種類用意

$$[a_+, a_+^\dagger] = 1,\ [a_+, a_+] = 0,\ [a_+^\dagger, a_+^\dagger] = 0$$

$$[a_-, a_-^\dagger] = 1,\ [a_-, a_-] = 0,\ [a_-^\dagger, a_-^\dagger] = 0$$

a_+とa_-は独立 ⇒ すべて可換

$$a = \begin{pmatrix} a_+ \\ a_- \end{pmatrix} \quad a^\dagger = (a_+^\dagger, a_-^\dagger) \quad \text{(spinor)}$$

$$\mathbf{J} \equiv a^\dagger \frac{1}{2}\boldsymbol{\sigma} a$$

$$J_x = a^\dagger \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} a$$

$$J_y = a^\dagger \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} a$$

$$J_z = a^\dagger \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} a$$

$$\begin{aligned}
[J_i, J_j] &= [a^\dagger \tfrac{1}{2}\sigma_i a,\ a^\dagger \tfrac{1}{2}\sigma_j a] \\
&= [a_\alpha^\dagger \tfrac{1}{2}(\sigma_i)_{\alpha\beta} a_\beta,\ a_\gamma^\dagger \tfrac{1}{2}(\sigma_j)_{\gamma\delta} a_\delta] \qquad \alpha, \beta : \pm \\
&= a_\alpha^\dagger \tfrac{1}{2}(\sigma_i)_{\alpha\beta}[a_\beta,\ a_\gamma^\dagger \tfrac{1}{2}(\sigma_j)_{\gamma\delta} a_\delta] \\
&\qquad + [a_\alpha^\dagger \tfrac{1}{2}(\sigma_i)_{\alpha\beta},\ a_\gamma^\dagger \tfrac{1}{2}(\sigma_j)_{\gamma\delta} a_\delta] a_\beta \\
&= a_\alpha^\dagger \tfrac{1}{2}(\sigma_i)_{\alpha\beta}\underbrace{[a_\beta, a_\gamma^\dagger]}_{=\delta_{\beta\gamma}}\tfrac{1}{2}(\sigma_j)_{\gamma\delta} a_\delta + a_\gamma^\dagger \tfrac{1}{2}(\sigma_j)_{\gamma\delta}\underbrace{[a_\alpha^\dagger, a_\delta]}_{=-\delta_{\alpha\delta}}\tfrac{1}{2}(\sigma_i)_{\alpha\beta} a_\beta \\
&= \tfrac{1}{4} a_\alpha^\dagger \underbrace{(\sigma_i)_{\alpha\beta}(\sigma_j)_{\beta\delta}}\, a_\delta - \tfrac{1}{4} a_\gamma^\dagger \underbrace{(\sigma_j)_{\gamma\alpha}(\sigma_i)_{\alpha\beta}}\, a_\beta \\
&= \tfrac{1}{4} a_\alpha^\dagger (\sigma_i\sigma_j)_{\alpha\delta} a_\delta - \tfrac{1}{4} a_\gamma^\dagger (\sigma_j\sigma_i)_{\gamma\beta} a_\beta \\
&= \tfrac{1}{4} a^\dagger (\sigma_i\sigma_j) a - \tfrac{1}{4} a^\dagger (\sigma_j\sigma_i) a \\
&= \tfrac{1}{4} a^\dagger (\sigma_i\sigma_j - \sigma_j\sigma_i) a = \tfrac{1}{2} a^\dagger \underbrace{(\sigma_i\sigma_j)}_{=i\varepsilon_{ijk}\sigma_k} a \\
&= i\varepsilon_{ijk}\underbrace{a^\dagger \tfrac{1}{2}\sigma_k a}_{=J_k} = i\varepsilon_{ij}J_k
\end{aligned}$$

$$\therefore [J_i, J_j] = i\varepsilon_{ijk}J_k$$

$\mathbf{J}$は角運動量の交換関係をみたす

$$\mathbf{J}^2 = \frac{1}{2}(J_+J_- + J_-J_+) + J_z^2$$

$$= \frac{1}{2}(a_+^\dagger a_- a_-^\dagger a_+ + a_-^\dagger a_+ a_+^\dagger a_-) + \frac{1}{4}(n_+ - n_-)^2$$

$\Big(J_\pm = J_x \pm iJ_y,\ J_+ = a_+^\dagger a_-,\ J_- = a_-^\dagger a_+$

$n_\pm = a_\pm^\dagger a_\pm,\ J_z = \frac{1}{2}(n_+ - n_-)$, a_+ と a_- は独立ですべて可換 $\Big)$

$$= \frac{1}{2}\Big(\underbrace{a_+^\dagger a_+}_{=n_+}\underbrace{a_- a_-^\dagger}_{=\underbrace{a_-^\dagger a_-}_{n_-}+1} + \underbrace{a_-^\dagger a_-}_{=n_-}\underbrace{a_+ a_+^\dagger}_{=\underbrace{a_+^\dagger a_+}_{=n_+}+1}\Big) + \frac{1}{4}(n_+ - n_-)^2$$

$$= \frac{1}{2}\{n_+(n_-+1) + n_-(n_++1)\} + \underbrace{\frac{1}{4}(n_+ - n_-)^2}_{=\frac{1}{4}(n_++n_-)^2 - n_+n_-}$$

$$= \frac{1}{2}(n_+ + n_-) + \frac{1}{4}(n_+ + n_-)^2$$

$n = n_+ + n_-$

$\mathbf{J}^2 = \frac{1}{4}n^2 + \frac{1}{2}n$, $m \equiv \frac{1}{2}(n_+ - n_-)$

$= \frac{1}{2}n\left(\frac{1}{2}n + 1\right)$ $\quad j = \frac{1}{2}(n_+ + n_-)$

$\left(j = \frac{1}{2}n\right)$ $\quad n_+ = j + m$

$= j(j+1)$ $\quad n_- = j - m$

$$|n_+, n_-\rangle = \frac{1}{\sqrt{n_+!}}(a_+^\dagger)^{n_+}\frac{1}{\sqrt{n_-!}}(a_-^\dagger)^{n_-}|0\rangle = \frac{(a_+^\dagger)^{n_+}(a_-^\dagger)^{n_-}}{\sqrt{n_+!\,n_-!}}|0\rangle$$

$\hat{n}_\pm = a_\pm^\dagger a_\pm$

$\hat{n}_+|n_+, n_-\rangle = n_+|n_+, n_-\rangle$

$\hat{n}_-|n_+ n_-\rangle = n_-|n_+ n_-\rangle$

$$|jm\rangle = |n_+ = j+m,\ n_- = j-m\rangle = \frac{(a_+)^{j+m}(a_-)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}|0\rangle$$

$\mathbf{J}^2|j,m\rangle = \hat{j}(\hat{j}+1)|j,m\rangle = j(j+1)|j,m\rangle$

$J_z|j,m\rangle = m|j,m\rangle$