量子力学3　7/5(金)

201110844
荻野孝浩

群の表現

対称操作 R　　$H \mapsto RHR^{-1} = H' = H$

$R^\dagger R = 1$　unitary　　$RH = HR$

$[H, R] = 0$

$H|\psi_j\rangle = |\psi_j\rangle E$

縮退度 d　$j = 1, 2, \cdots, d$

(縮退したすべてを含む)

$R|\psi_j\rangle$

$HR|\psi_j\rangle = RH|\psi_j\rangle = R|\psi_j\rangle E$

$R|\psi_j\rangle$: $\{|\psi_1\rangle, \cdots, |\psi_d\rangle\}$ の線型結合

$R|\psi_j\rangle = \sum_i |\psi_i\rangle D_{ij}(R) = (\psi D)_j$

← 線型結合の係数

$\psi(|\psi_1\rangle, |\psi_2\rangle, \cdots, |\psi_d\rangle)$

$(R|\psi_1\rangle, R|\psi_2\rangle, \cdots, R|\psi_d\rangle)$
$= R(|\psi_1\rangle, |\psi_2\rangle, \cdots, |\psi_d\rangle)$
$= R\psi$
$= ((\psi D)_1, (\psi D)_2, \cdots, (\psi D)_d) = \psi D(R)$

$R\psi = \psi D(R)$

$R(|\psi_1\rangle, \cdots, |\psi_d\rangle) = (|\psi_1\rangle, \cdots, |\psi_d\rangle) D(R)$

対称操作の集合 $\{R\}$、群をつくる　$R_2 R_1 = {}^\exists R_{21}$

$$R_1\psi = \psi D(R_1)$$
$$\underbrace{R_2R_1}_{= R_{21}}\psi = R_2\psi D(R_1)$$
$$= \psi D(R_2) D(R_1)$$
$$= R_{21}\psi = \psi D(R_{21})$$

$$\therefore\ D(R_2) D(R_1) = D(R_{21})$$

$$D(e) = E_d$$

$$D(R^{-1}) = [D(R)]^{-1}$$

$D(R)$: 群$\{R\}$の行列による表現という
R : unitary

$R\psi = \psi D$, $\psi^\dagger R^\dagger = D^\dagger \psi^\dagger \to \psi^\dagger R^\dagger R \psi = E_d$
$= D^\dagger \psi^\dagger \psi D$
$= D^\dagger D$
∴ D : unitary行列

$\psi = (|\psi_1\rangle, |\psi_2\rangle, \cdots, |\psi_d\rangle)$

$$\psi^\dagger \psi = \begin{pmatrix} \langle \psi_1 | \\ \vdots \\ \langle \psi_d | \end{pmatrix} (|\psi_1\rangle, \cdots, |\psi_d\rangle)$$

$$= \begin{pmatrix} \langle \psi_1 | \psi_1 \rangle & \cdots & \langle \psi_1 | \psi_d \rangle \\ \vdots & \ddots & \vdots \\ \langle \psi_d | \psi_1 \rangle & \cdots & \langle \psi_d | \psi_d \rangle \end{pmatrix}$$

規格直交
$\langle \psi_i, \psi_j \rangle = \delta_{ij}$

$= E_d$

☆ 回転群の表現 ($\hbar = 1$ 自然単位)

$R_\alpha(\hat{n}) = e^{-i\alpha \hat{n}\cdot\vec{J}}$ $[J_i, J_j] = i\varepsilon_{ijk} J_k$
↓
$D(R)$: 行列

無限小変換 $\alpha = \delta\alpha$, $e^{-i\delta\alpha \hat{n}\cdot\vec{J}} = 1 - i\delta\alpha\, \hat{n}\cdot\vec{J}$

$R\psi = \psi D$
$$(1 - i\delta\alpha\, \hat{n}\cdot\vec{J})\psi = \psi \underbrace{D(1 - i\delta\alpha\, \hat{n}\cdot\vec{J})}_{1 - i\delta\alpha D(\hat{n}\cdot\vec{J})}$$

$(\hat{n}\cdot\vec{J})\psi = \psi D(\hat{n}\cdot\vec{J})$ ← が決まれば表現が定まる。

$J_x J_y J_z \to J_z$ J_zを考えれば十分
$J_z \psi = \psi D(J_z)$
$J_z(|\psi_1\rangle, \cdots, |\psi_d\rangle) = (|\psi_1\rangle, \cdots, |\psi_d\rangle) D(J_z)$
$$J_z(\underbrace{|\psi_{-j}\rangle, |\psi_{-j+1}\rangle, \cdots, |\psi_m\rangle, \cdots, |\psi_j\rangle}_{2j+1\text{個}})$$

$$= (\cdots |\psi_m\rangle \cdots) \begin{pmatrix} -\hbar j & & & & 0 \\ & \ddots & & & \\ & & \hbar m & & \\ & & & \ddots & \\ 0 & & & & \hbar j \end{pmatrix}$$

$J_z|\psi_m\rangle = m|\psi_m\rangle \quad (\hbar = 1)$ $|\psi_m\rangle$：J_zの固有状態

$= |\psi_{m'}\rangle D^j_{m'm}(J_z)$

$D^j_{m'm}(J_z) = m\,\delta_{m'm} \quad \cdots$ ※1

$J_\pm|\psi_m\rangle = \sqrt{(j\mp m)(j\pm m+1)}\,|\psi_{m\pm 1}\rangle$

$= |\psi_{m'}\rangle\ D^j_{m'm}(J_\pm)$

$D^j_{m'm}(J_\pm) = \sqrt{(j\mp m)(j\pm m+1)}\ \delta_{m'm}$

$\cdots$ ※2

※1 ※2 で定まる行列

$D^j(J_z)$、$D^j(J_\pm)$ ：$2j+1$ 次元行列

spin-j 表現

一般に Eulan角(α, β, γ)で定まる

$R = R_\alpha(z) R_\beta(y) R_\gamma(z) = e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z}$

$D^j(R(\alpha\beta\gamma)) = D^j(e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z})$

$= D^j(e^{-i\alpha J_z}) D^j(e^{-i\beta J_y}) D^j(e^{-i\gamma J_z})$

$|\psi_m\rangle = |jm\rangle \qquad m = -j, \cdots, j$

$D^j_{m'm}(R) = \langle jm'|R|jm\rangle$

$= \underbrace{\langle jm'|e^{-i\alpha J_z}}_{e^{-i\alpha m'}\langle jm'|} e^{-i\beta J_y} \underbrace{e^{-i\gamma J_z}|jm\rangle}_{e^{-i\gamma m}|jm\rangle}$

$= e^{-i\alpha m'}\langle jm'|e^{-i\beta J_y}|jm\rangle e^{-i\gamma m}$

$D^j_{m'm}(R(\alpha\beta\gamma)) = e^{-i\alpha m'}\ d^j_{m'm}\ e^{-i\gamma m}$

$\langle jm'|e^{-i\beta J_y}|jm\rangle$

Schwinger Boson を用いた$d^j_{m'm}$の決定

Boson

$[a, a^+] = 1,\ [a, a] = 0,\ [a^+, a^+] = 0$

$a|0\rangle = 0 \qquad |n\rangle = \frac{1}{\sqrt{n}}(a^+)^n|0\rangle$

$a^+a|n\rangle = n|n\rangle$

$a^+a = n$ ：粒子数の演算子

+、−の2種類用意

$[a_+, a_+^\dagger] = 1$ 、 $[a_+, a_+] = 0$ 、 $[a_+^\dagger, a_+^\dagger] = 0$

$[a_-, a_-^\dagger] = 1$ 、 $[a_-, a_-] = 0$ 、 $[a_-^\dagger, a_-^\dagger] = 0$

a_+ と a_- は独立
すべて可換

$$a = \begin{pmatrix} a_+ \\ a_- \end{pmatrix} \quad a^\dagger = (a_+^\dagger, a_-^\dagger) \quad (\text{spinor})$$

$\vec{J} \equiv a^\dagger \frac{1}{2} \vec{\sigma} a$ とせよ

$$J_x = a^\dagger \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} a$$

$$J_y = a^\dagger \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} a$$

$$J_z = a^\dagger \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} a$$

$$\begin{aligned}
[J_i, J_j] &= [\, a^\dagger \tfrac{1}{2} \sigma_i a \,,\ a^\dagger \tfrac{1}{2} \sigma_j a \,] \\
&= [\, a_\alpha^\dagger \tfrac{1}{2} (\sigma_i)_{\alpha\beta} a_\beta ,\ a_\gamma^\dagger \tfrac{1}{2} (\sigma_j)_{\gamma\delta} a_\delta \,] \\
&= a_\alpha^\dagger \tfrac{1}{2} (\sigma_i)_{\alpha\beta} [\, a_\beta ,\ a_\gamma^\dagger \tfrac{1}{2} (\sigma_j)_{\gamma\delta} a_\delta \,] \\
&\quad + [\, a_\alpha^\dagger \tfrac{1}{2} (\sigma_i)_{\alpha\beta} ,\ a_\gamma^\dagger \tfrac{1}{2} (\sigma_j)_{\gamma\delta} a_\delta \,]\, a_\beta \\
&= a_\gamma^\dagger \tfrac{1}{2} (\sigma_i)_{\alpha\beta} [a_\beta, a_\gamma^\dagger] \tfrac{1}{2} (\sigma_j)_{\gamma\delta} a_\delta \\
&\quad + a_\gamma^\dagger \tfrac{1}{2} (\sigma_j)_{\gamma\delta} \underbrace{[a_\alpha^\dagger, a_\delta]}_{= -\delta_{\alpha\delta}} \tfrac{1}{2} (\sigma_i)_{\alpha\beta} a_\beta \qquad ([a_\beta, a_\gamma] = 0) \\
&= \tfrac{1}{4} a_\alpha^\dagger (\sigma_i)_{\alpha\beta} (\sigma_j)_{\beta\delta} a_\delta \\
&\quad - \tfrac{1}{4} a_\gamma^\dagger (\sigma_j)_{\gamma\delta} (\sigma_i)_{\delta\beta} a_\beta \\
&= \tfrac{1}{4} a_\alpha^\dagger (\sigma_i \sigma_j)_{\beta\delta} a_\delta - \tfrac{1}{4} a_\gamma^\dagger (\sigma_j \sigma_i)_{\gamma\beta} a_\beta \\
&= \tfrac{1}{4} a^\dagger (\sigma_i \sigma_j) a - \tfrac{1}{4} a^\dagger (\sigma_j \sigma_i) a \\
&= \tfrac{1}{4} a^\dagger (\sigma_i \sigma_j - \sigma_j \sigma_i) a = \tfrac{1}{2} a^\dagger (\sigma_i \sigma_j) a
\end{aligned}$$

$$\because \ [J_i, J_j] = a^\dagger \frac{1}{2} \underbrace{(\sigma_i \sigma_j)}_{i\varepsilon_{ijk}\sigma_k} a$$

$$= i\varepsilon_{ijk} \underbrace{a^\dagger \frac{1}{2}\sigma_k a}_{J_k} \quad = i\varepsilon_{ijk} J_k$$

$\vec{J}$ は角運動量の交換関係をみたす.

$$\vec{J}^2 = \frac{1}{2}(J_+ J_- + J_- J_+) + J_z^2$$

$$= \frac{1}{2}(a_+^\dagger a_- a_-^\dagger a_+ + a_-^\dagger a_+ a_+^\dagger a_-) + \frac{1}{4}(n_+ - n_-)^2$$

$$= \frac{1}{2}(\underbrace{a_+^\dagger a_+}_{n_+} \underbrace{a_- a_-^\dagger}_{\underbrace{a_-^\dagger a_-}_{n_-} + 1} + \underbrace{a_-^\dagger a_-}_{n_-} \underbrace{a_+ a_+^\dagger}_{\underbrace{a_+^\dagger a_+}_{n_+} + 1}) + \frac{1}{4}(n_+ - n_-)^2$$

$$\left(\begin{array}{ll} J_\pm = J_x \pm i J_y & \\ J_+ = a_+^\dagger a_- & n_\pm = a_\pm^\dagger a_\pm \\ J_- = a_-^\dagger a_+ & J_z = \frac{1}{2}(n_+ - n_-) \end{array}\right)$$

$$\vec{J}^2 = \frac{1}{2}\{n_+(n_- + 1) + n_-(n_+ + 1)\} + \underbrace{\frac{1}{4}(n_+ - n_-)^2}_{\frac{1}{4}(n_+ + n_-)^2 - n_+ n_-}$$

$$= \frac{1}{2}(n_+ + n_-) + \frac{1}{4}(n_+ + n_-)^2$$

$n = n_+ + n_-$ として

$$\vec{J}^2 = \frac{1}{2}n + \frac{1}{4}n^2 = \frac{1}{2}n\left(\frac{1}{2}n + 1\right)$$

$$J = \frac{1}{2}n \qquad \vec{J}^2 = J(J+1)$$

$$m \equiv \frac{1}{2}(n_+ - n_-) \qquad n_+ = j + m$$
$$j = \frac{1}{2}(n_+ + n_-) \qquad n_- = j - m$$

$$|n_+ n_-\rangle = \frac{1}{\sqrt{n_+!}}(a_+^\dagger)^{n_+}\frac{1}{\sqrt{n_-!}}(a_-^\dagger)^{n_-}|0\rangle$$
$$= \frac{(a_+^\dagger)^{n_+}(a_-^\dagger)^{n_-}}{\sqrt{n_+!\,n_-!}}|0\rangle$$

$$\hat{n}_\pm = a_\pm^\dagger a_\pm$$

$$\hat{n}_+|n_+ n_-\rangle = n_+|n_+ n_-\rangle,\quad \hat{n}_-|n_+ n_-\rangle = n_-|n_+ n_-\rangle$$

$$|jm\rangle = |n_+ = j+m,\ n_- = j-m\rangle$$
$$= \frac{(a_+^\dagger)^{j+m}(a_-^\dagger)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}|0\rangle$$

$$\vec{J}^2|jm\rangle = \hat{j}(\hat{j}+1)|jm\rangle = j(j+1)|jm\rangle$$

$$J_z|jm\rangle = m|jm\rangle \qquad J_z = \frac{1}{2}(n_1 - n_2) = \hat{n}$$

$$\langle jm'|e^{-i\beta J_y}|jm\rangle = d^j_{m'm}$$