201110878 平原 亮磨

問題1. 2つの独立な角運動量を $\vec{J}_1, \vec{J}_2$

$[J_{i\mu}, J_{j\nu}] = \delta_{ij} i\hbar \varepsilon_{\mu\nu\lambda} J_{i\lambda}$, $i = 1, 2$ とし、状態 $|j_i m_i\rangle$ を

$\vec{J}_i^2 |j_i m_i\rangle = \hbar^2 j_i(j_i+1)|j_i m_i\rangle$, $J_{iz}|j_i m_i\rangle = \hbar m_i |j_i m_i\rangle$ とする。

また、以下の公式を用いよ。

$$\begin{cases} |j\ m-1\rangle = \dfrac{1}{\hbar\sqrt{(j+m)(j-m+1)}} J_-|j\ m\rangle \\ |j\ m-2\rangle = \dfrac{1}{\hbar^2\sqrt{(j+m)(j+m-1)\cdot(j-m+1)(j-m+2)}} J_-^2|j\ m\rangle \\ \vdots \\ |j\ m-k\rangle = \hbar^{-k}\left[(j+m)\cdots(j+m-k+1)\cdot(j-m+1)\cdots(j-m+k)\right]^{-\frac{1}{2}} J_-^k|j\ m\rangle \end{cases}$$

(1) $\vec{J} = \vec{J}_1 + \vec{J}_2$ とするとき、$\vec{J}$ が角運動量の代数を満たすことを示せ。

$$\begin{aligned} [J_\mu, J_\nu] &= [J_{1\mu} + J_{2\mu}, J_{1\nu} + J_{2\nu}] \\ &= [J_{1\mu}, J_{1\nu}] + [J_{2\mu}, J_{2\nu}] \qquad (\because [J_{1\mu}, J_{2\nu}] = 0) \\ &= i\hbar \varepsilon_{\mu\nu\lambda} J_{1\lambda} + i\hbar \varepsilon_{\mu\nu\lambda} J_{2\lambda} \\ &= i\hbar \varepsilon_{\mu\nu\lambda} \underbrace{(J_{1\lambda} + J_{2\lambda})}_{J_\lambda} \\ &= i\hbar \varepsilon_{\mu\nu\lambda} J_\lambda \end{aligned}$$

(2) $j_{max} = j_1 + j_2$, $j_{min} = |j_1 - j_2|$ として、$\displaystyle\sum_{j=j_{min}}^{j_{max}} (2j+1) = (2j_1+1)(2j_2+1)$ を示せ。

$$\sum_{j=j_{min}}^{j_{max}}(2j+1) = 2\underbrace{\sum_{j=j_{min}}^{j_{max}} j}_{①} + \underbrace{\sum_{j=j_{min}}^{j_{max}} 1}_{②}$$

ここで、aを初項，dを公差，nを項数とすると、

$$① = \sum_{j=j_{min}}^{j_{max}} j = \frac{1}{2}n\left(2a + (n-1)d\right)$$

$$= \frac{1}{2}(j_{max} - j_{min} + 1)\left(2 \cdot j_{min} + (j_{max} - j_{min} + 1 - 1) \cdot 1\right)$$

$$= \frac{1}{2}(j_{max} - j_{min} + 1)(j_{max} + j_{min})$$

$$② = \sum_{j=j_{min}}^{j_{max}} 1 = n = j_{max} - j_{min} + 1$$

よって、$\displaystyle\sum_{j=j_{min}}^{j_{max}}(2j+1) = 2 \cdot \frac{1}{2}(j_{max} - j_{min} + 1)(j_{max} + j_{min}) + (j_{max} - j_{min} + 1)$

$$= (j_{max} - j_{min} + 1)(j_{max} + j_{min} + 1)$$

○ $j_1 > j_2$ として、$j_{max} = j_1 + j_2$，$j_{min} = j_1 - j_2$ を代入すると、

$$= \left(j_1 + j_2 - (j_1 - j_2) + 1\right)\left(j_1 + j_2 + j_1 - j_2 + 1\right)$$

$$= (2j_1 + 1)(2j_2 + 1)$$

(3) Clebsh-Gordan 係数の直交性を示せ。

Clebsh-Gordan 係数は、$\langle j'm' | j_1 m_1 j_2 m_2\rangle\langle j_1 m_1 j_2 m_2 | jm\rangle$ ——(A)

$\langle j_1 m_1' j_2 m_2' | jm\rangle\langle jm | j_1 m_1 j_2 m_2\rangle$ ——(B)

$$(A) = \langle j'm'| \Big(\underbrace{|j_1 m_1\rangle\langle j_1 m_1|}_{=1} \otimes \underbrace{|j_2 m_2\rangle\langle j_2 m_2|}_{=1} \Big) |jm\rangle$$

$$= \langle j'm' | jm\rangle = \delta_{jj'}\delta_{mm'}$$

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$$(B) = \langle j_1 m_1' j_2 m_2' | jm\rangle \underbrace{\langle jm | j_1 m_1 j_2 m_2\rangle}_{=1}$$

$$= \langle j_1 m_1' j_2 m_2' | j_1 m_1 j_2 m_2\rangle$$

$$= \underbrace{\langle j_1 m_1' | j_1 m_1\rangle}_{=\delta_{m_1 m_1'}} \otimes \underbrace{\langle j_2 m_2' | j_2 m_2\rangle}_{=\delta_{m_2 m_2'}} = \underline{\delta_{m_1 m_1'} \delta_{m_2 m_2'}}$$

問題 2 $j_1 = \frac{1}{2}$, $j_2 = \frac{1}{2}$ のClebsch-Gordan係数を求めよう。

(1) $J_z \left|\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle$ と $J_+ \left|\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle$ を計算し、$\left|\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle$ が $|11\rangle$ であることを示せ。

$$J_z \left|\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right\rangle = (J_{1z} + J_{2z}) \left|\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right\rangle$$

$$= \underbrace{J_{1z}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle}_{=\frac{\hbar}{2}\left|\frac{1}{2},\frac{1}{2}\right\rangle}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle + \left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle \underbrace{J_{2z}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle}_{=\frac{\hbar}{2}\left|\frac{1}{2},\frac{1}{2}\right\rangle}$$

$$= \frac{\hbar}{2}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle + \frac{\hbar}{2}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle$$

$$= \hbar \cdot \underbrace{1}_{=m} \left|\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right\rangle \qquad \therefore m = 1$$

$$J_+ \left|\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right\rangle = (J_{1+} + J_{2+}) \left|\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right\rangle$$

$$= \underbrace{J_{1+}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle}_{=\hbar\sqrt{\left(\frac{1}{2}-\frac{1}{2}\right)\left(\frac{1}{2}+\frac{1}{2}+1\right)}\left|\frac{1}{2},\frac{3}{2}\right\rangle = 0}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle + \left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle \underbrace{J_{2+}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle}_{=\hbar\sqrt{\left(\frac{1}{2}-\frac{1}{2}\right)\left(\frac{1}{2}+\frac{1}{2}+1\right)}\left|\frac{1}{2},\frac{3}{2}\right\rangle = 0}$$

$$= 0 \qquad \therefore j = m$$

$$\therefore \left|\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right\rangle = \underline{|1,1\rangle}$$

(2) $|10\rangle \propto J_-|11\rangle$ を求めよ。

$$|10\rangle \propto J_-|11\rangle$$

$$= (J_{1-}+J_{2-})\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle$$

$$= \underbrace{J_{1-}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle}_{=\hbar\sqrt{\left(\frac{1}{2}+\frac{1}{2}\right)\left(\frac{1}{2}-\frac{1}{2}+1\right)}\left|\frac{1}{2},-\frac{1}{2}\right\rangle}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle + \left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle\underbrace{J_{2-}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle}_{=\hbar\sqrt{\left(\frac{1}{2}+\frac{1}{2}\right)\left(\frac{1}{2}-\frac{1}{2}+1\right)}\left|\frac{1}{2},-\frac{1}{2}\right\rangle}$$

$$= \hbar\left|\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right\rangle + \hbar\left|\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}\right\rangle$$

$$\therefore\ |10\rangle \propto \hbar\left(\left|\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right\rangle + \left|\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}\right\rangle\right)$$

(3) $|1\,{-1}\rangle \propto J_-^2|11\rangle$ を求めよ。

$$|1\,{-1}\rangle \propto J_-^2|1,1\rangle$$

$$= (J_{1-}+J_{2-})^2|1,1\rangle = (J_{1-}^2+2J_{1-}J_{2-}+J_{2-}^2)\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle$$

$$= \underbrace{J_{1-}^2\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle}_{\substack{=\hbar^2\sqrt{\left(\frac{1}{2}+\frac{1}{2}\right)\left(\frac{1}{2}+\frac{1}{2}-1\right)\left(\frac{1}{2}-\frac{1}{2}+1\right)\left(\frac{1}{2}-\frac{1}{2}+2\right)}\left|\frac{1}{2},-\frac{3}{2}\right\rangle \\ =0}}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle + 2J_{1-}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle J_{2-}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle + \left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle\underbrace{J_{2-}^2\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle}_{=0}$$

$$= 2\hbar\left|\tfrac{1}{2},\tfrac{-1}{2}\right\rangle\hbar\left|\tfrac{1}{2},\tfrac{-1}{2}\right\rangle$$

$$= 2\hbar^2\left|\tfrac{1}{2},\tfrac{-1}{2},\tfrac{1}{2},\tfrac{-1}{2}\right\rangle$$

$$\therefore\ |1\,{-1}\rangle \propto 2\hbar^2\left|\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}\right\rangle$$

(4) $|00\rangle = a\left|\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right\rangle + b\left|\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}\right\rangle$ として、$\langle 10|00\rangle = 0$ を満たすように a, b を定めることで求めよ。但し、$b>0$ とせよ。なお $\langle 00|00\rangle = |a|^2+|b|^2 = 1$ である。

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$$J_-|11\rangle = \hbar\sqrt{(1+1)(1-1+1)}\,|10\rangle$$

$= \hbar\sqrt{2}\,|10\rangle$ だから、(2) の結果を用いて、

$$|10\rangle = \frac{1}{\sqrt{2}}\left(\left|\tfrac{1}{2}\,-\tfrac{1}{2},\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle + \left|\tfrac{1}{2}\,\tfrac{1}{2},\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle\right)$$

$$\langle 10| = \frac{1}{\sqrt{2}}\left(\left\langle\tfrac{1}{2}\,-\tfrac{1}{2},\tfrac{1}{2}\,\tfrac{1}{2}\right| + \left\langle\tfrac{1}{2}\,\tfrac{1}{2},\tfrac{1}{2}\,-\tfrac{1}{2}\right|\right)$$ だから、

$$\langle 10|00\rangle = \frac{1}{\sqrt{2}}\left(\left\langle\tfrac{1}{2}\,-\tfrac{1}{2},\tfrac{1}{2}\,\tfrac{1}{2}\right| + \left\langle\tfrac{1}{2}\,\tfrac{1}{2},\tfrac{1}{2}\,-\tfrac{1}{2}\right|\right)\left(a\left|\tfrac{1}{2}\,-\tfrac{1}{2},\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle + b\left|\tfrac{1}{2}\,\tfrac{1}{2},\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\Bigg(a\underbrace{\left\langle\tfrac{1}{2}\,-\tfrac{1}{2},\tfrac{1}{2}\,\tfrac{1}{2}\middle|\tfrac{1}{2}\,-\tfrac{1}{2},\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle}_{=1} + b\underbrace{\left\langle\tfrac{1}{2}\,-\tfrac{1}{2},\tfrac{1}{2}\,\tfrac{1}{2}\middle|\tfrac{1}{2}\,\tfrac{1}{2},\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle}_{=0}$$

$$+ a\underbrace{\left\langle\tfrac{1}{2}\,\tfrac{1}{2},\tfrac{1}{2}\,-\tfrac{1}{2}\middle|\tfrac{1}{2}\,-\tfrac{1}{2},\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle}_{=0} + b\underbrace{\left\langle\tfrac{1}{2}\,\tfrac{1}{2},\tfrac{1}{2}\,-\tfrac{1}{2}\middle|\tfrac{1}{2}\,\tfrac{1}{2},\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle}_{=1}\Bigg)$$

$$= \frac{1}{\sqrt{2}}(a+b) = 0$$

$\therefore\ a + b = 0$

一方、$|a|^2 + |b|^2 = 1$ だから $a = -b$ を代入して、

$$2|b|^2 = 1$$

$$|b| = \pm\frac{1}{\sqrt{2}} \implies b = \frac{1}{\sqrt{2}} \quad (\because b > 0)$$ //

$$\therefore\ a = -\frac{1}{\sqrt{2}}$$ //

(5) ここで求めた $|00\rangle$ が $J_+|00\rangle = 0$ を満たすことを確認せよ。

$$|00\rangle = -\frac{1}{\sqrt{2}}\left|\tfrac{1}{2}\,-\tfrac{1}{2},\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle + \frac{1}{\sqrt{2}}\left|\tfrac{1}{2}\,\tfrac{1}{2},\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle$$

$$= \frac{1}{\sqrt{2}}\left(-\left|\tfrac{1}{2}\,-\tfrac{1}{2},\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle + \left|\tfrac{1}{2}\,\tfrac{1}{2},\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle\right)$$

$$J_+|00\rangle = (J_{1+}+J_{2+})\frac{1}{\sqrt{2}}\left(-\left|\tfrac12\,-\tfrac12,\tfrac12\,\tfrac12\right\rangle + \left|\tfrac12\,\tfrac12,\tfrac12\,-\tfrac12\right\rangle\right)$$

$$=\frac{1}{\sqrt{2}}\Big\{-\underbrace{J_{1+}\left|\tfrac12\,-\tfrac12\right\rangle}_{=\hbar\sqrt{(\frac12+\frac12)(\frac12-\frac12+1)}\,|\frac12\,\frac12\rangle}\left|\tfrac12\,\tfrac12\right\rangle + \underbrace{J_{1+}\left|\tfrac12\,\tfrac12\right\rangle}_{=0}\left|\tfrac12\,-\tfrac12\right\rangle - \left|\tfrac12\,-\tfrac12\right\rangle\underbrace{J_{2+}\left|\tfrac12\,\tfrac12\right\rangle}_{=0}$$

$$+\left|\tfrac12\,\tfrac12\right\rangle\underbrace{J_{2+}\left|\tfrac12\,-\tfrac12\right\rangle}_{=\hbar\sqrt{(\frac12+\frac12)(\frac12-\frac12+1)}\,|\frac12\,\frac12\rangle}\Big\}$$

$$=\frac{1}{\sqrt{2}}\left\{-\hbar\left|\tfrac12\,\tfrac12,\tfrac12\,\tfrac12\right\rangle+\hbar\left|\tfrac12\,\tfrac12,\tfrac12\,\tfrac12\right\rangle\right\}$$

$$=0$$

(6) $\Psi_0 = (|10\rangle, |00\rangle)$, $\psi_0 = \left(\left|\tfrac12\,-\tfrac12,\tfrac12\,\tfrac12\right\rangle, \left|\tfrac12\,\tfrac12,\tfrac12\,-\tfrac12\right\rangle\right)$ として $|10\rangle = \psi_0\psi_c$ を満たす二成分の列ベクトル ψ_c を求めよ。

$$|10\rangle = \frac{1}{\sqrt{2}}\left|\tfrac12\,-\tfrac12,\tfrac12\,\tfrac12\right\rangle + \frac{1}{\sqrt{2}}\left|\tfrac12\,\tfrac12,\tfrac12\,-\tfrac12\right\rangle$$

$$=\left(\left|\tfrac12\,-\tfrac12,\tfrac12\,\tfrac12\right\rangle, \left|\tfrac12\,\tfrac12,\tfrac12\,-\tfrac12\right\rangle\right)\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$$

$$\therefore\ \psi_c = \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$$

(7) 2行2列の行列 $P=\psi_c\psi_c^{\dagger}$ を求めよ。

$$P = \psi_c\psi_c^{\dagger} = \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix} = \begin{pmatrix}1/2 & 1/2\\1/2 & 1/2\end{pmatrix}$$

(8) $(E_2-P)\psi_c=0$ を示せ。ここで E_2 は2次元単位行列である。

$$(E_2-P)\psi_c = \left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}-\begin{pmatrix}\frac12&\frac12\\\frac12&\frac12\end{pmatrix}\right\}\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$$

$$=\begin{pmatrix}\frac12&-\frac12\\-\frac12&\frac12\end{pmatrix}\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{2\sqrt{2}}-\frac{1}{2\sqrt{2}}\\-\frac{1}{2\sqrt{2}}+\frac{1}{2\sqrt{2}}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}=0$$

(9) $\phi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ として $\psi' = (E_2 - P)\phi$ を求めよ。

$$\psi' = (E_2 - P)\phi$$

$$= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

(10) $\psi = \psi' / \|\psi'\|$ として $|00\rangle = \Psi_0 \psi$ であることを確認せよ。

$$\|\psi'\| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{4}} = \frac{1}{\sqrt{2}}$$

$$\therefore \ \psi = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$\Psi_0 \psi = \left(\left|\tfrac{1}{2} -\tfrac{1}{2}, \tfrac{1}{2} \tfrac{1}{2}\right\rangle, \left|\tfrac{1}{2} \tfrac{1}{2}, \tfrac{1}{2} -\tfrac{1}{2}\right\rangle \right) \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$= \left(-\frac{1}{\sqrt{2}} \left|\tfrac{1}{2} -\tfrac{1}{2}, \tfrac{1}{2} \tfrac{1}{2}\right\rangle + \frac{1}{\sqrt{2}} \left|\tfrac{1}{2} \tfrac{1}{2}, \tfrac{1}{2} -\tfrac{1}{2}\right\rangle \right) = |00\rangle$$

(11) 次の2行2列の行列 M を求めよ。

$$\Phi_0 = \Psi_0 M$$

$$(|10\rangle, |00\rangle) = \left(\left|\tfrac{1}{2} -\tfrac{1}{2}, \tfrac{1}{2} \tfrac{1}{2}\right\rangle, \left|\tfrac{1}{2} \tfrac{1}{2}, \tfrac{1}{2} -\tfrac{1}{2}\right\rangle \right) M$$

$$\begin{cases} \circ \ |10\rangle = \frac{1}{\sqrt{2}} \left|\tfrac{1}{2} -\tfrac{1}{2}, \tfrac{1}{2} \tfrac{1}{2}\right\rangle + \frac{1}{\sqrt{2}} \left|\tfrac{1}{2} \tfrac{1}{2}, \tfrac{1}{2} -\tfrac{1}{2}\right\rangle \\ \circ \ |00\rangle = -\frac{1}{\sqrt{2}} \left|\tfrac{1}{2} -\tfrac{1}{2}, \tfrac{1}{2} \tfrac{1}{2}\right\rangle + \frac{1}{\sqrt{2}} \left|\tfrac{1}{2} \tfrac{1}{2}, \tfrac{1}{2} -\tfrac{1}{2}\right\rangle \end{cases}$$ だから、

$$(|10\rangle, |00\rangle) = \left(\left|\tfrac{1}{2} -\tfrac{1}{2}, \tfrac{1}{2} \tfrac{1}{2}\right\rangle, \left|\tfrac{1}{2} \tfrac{1}{2}, \tfrac{1}{2} -\tfrac{1}{2}\right\rangle \right) \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$\therefore \ M = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

問題3 $j_1 = 1$, $j_2 = 1$ の Clebsch-Gordan 係数を求めよう。

(1) $J_z |1111\rangle$ と $J_+ |1111\rangle$ を計算し $|1111\rangle$ が $|22\rangle$ であることを示せ。

$$J_z |1111\rangle = (J_{1z} + J_{2z})|11\rangle|11\rangle$$

$$= J_{1z}|11\rangle|11\rangle + |11\rangle J_{2z}|11\rangle$$

$$= \hbar|11\rangle|11\rangle + |11\rangle \cdot \hbar|11\rangle$$

$$= \hbar \cdot \underbrace{2}_{=m} |1111\rangle \qquad \therefore\ m = 2$$

$$J_+ |1111\rangle = (J_{1+} + J_{2+})|11\rangle|11\rangle$$

$$= \underbrace{J_{1+}|11\rangle}_{=\hbar\sqrt{(1-1)(1+1+1)}|12\rangle\ =0}|11\rangle + |11\rangle \underbrace{J_{2+}|11\rangle}_{=\hbar\sqrt{(1-1)(1+1+1)}|12\rangle\ =0}$$

$$= 0 \qquad \therefore\ j = m$$

以上より、$|22\rangle = |1111\rangle$

(2) $|21\rangle$ を求めよ。

$$J_- |22\rangle = (J_{1-} + J_{2-})|11\rangle|11\rangle$$

$$= \underbrace{J_{1-}|11\rangle}_{=\hbar\sqrt{(1+1)(1-1+1)}|10\rangle}|11\rangle + |11\rangle \underbrace{J_{2-}|11\rangle}_{=\hbar\sqrt{(1+1)(1-1+1)}|10\rangle}$$

$$= \hbar\sqrt{2}\left(|1011\rangle + |1110\rangle\right)$$

一方, $J_- |22\rangle = \hbar\sqrt{(2+2)(2-2+1)}\,|21\rangle$

$$= \hbar \cdot 2\,|21\rangle$$

$$\therefore\ |21\rangle = \frac{1}{\sqrt{2}}\left(|1011\rangle + |1110\rangle\right)$$

(3) $|20\rangle$ を求めよ。

$$J_-|21\rangle = (J_{1-} + J_{2-})\cdot\frac{1}{\sqrt{2}}\left(|1011\rangle + |1110\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\Big\{ \underbrace{J_{1-}|10\rangle}_{=\hbar\sqrt{1\cdot 2}|1-1\rangle}|11\rangle + \underbrace{J_{1-}|11\rangle}_{=\hbar\sqrt{2\cdot 1}|10\rangle}|10\rangle + |10\rangle \underbrace{J_{2-}|11\rangle}_{=\hbar\sqrt{2\cdot 1}|10\rangle} + |11\rangle \underbrace{J_{2-}|10\rangle}_{=\hbar\sqrt{1\cdot 2}|1-1\rangle} \Big\}$$

$$= \hbar\left(|1-111\rangle + |1010\rangle + |1010\rangle + |111-1\rangle\right)$$

$$= \hbar\left(|1-111\rangle + 2|1010\rangle + |111-1\rangle\right)$$

一方, $J_-|21\rangle = \hbar\sqrt{(2+1)(2-1+1)}\,|20\rangle$

$$= \hbar\sqrt{6}\,|20\rangle$$

$$\therefore\ |20\rangle = \frac{1}{\sqrt{6}}\left(|1-111\rangle + 2|1010\rangle + |111-1\rangle\right)$$ //

(4) $|2-1\rangle$ を求めよ。

$$J_-|20\rangle = (J_{1-} + J_{2-})\frac{1}{\sqrt{6}}\left(|1-111\rangle + 2|1010\rangle + |111-1\rangle\right)$$

$$= \frac{1}{\sqrt{6}}\Big\{ \underbrace{J_{1-}|1-1\rangle}_{=\hbar\sqrt{0\cdot 1}|1-2\rangle}|11\rangle + 2\underbrace{J_{1-}|10\rangle}_{=\hbar\sqrt{1\cdot 2}|1-1\rangle}|10\rangle + \underbrace{J_{1-}|11\rangle}_{=\hbar\sqrt{2\cdot 1}|10\rangle}|1-1\rangle$$

$$+ |1-1\rangle \underbrace{J_{2-}|11\rangle}_{=\hbar\sqrt{2\cdot 1}|10\rangle} + 2|10\rangle \underbrace{J_{2-}|10\rangle}_{=\hbar\sqrt{1\cdot 2}|1-1\rangle} + |11\rangle \underbrace{J_{2-}|1-1\rangle}_{=\hbar\sqrt{0\cdot 1}|1-2\rangle} \Big\}$$

$$= \frac{\hbar}{\sqrt{3}}\left\{ 2|1-110\rangle + |101-1\rangle + |1-110\rangle + 2|101-1\rangle \right\}$$

$$= \hbar\sqrt{3}\left(|1-110\rangle + |101-1\rangle\right)$$

一方, $J_-|20\rangle = \hbar\sqrt{(2+0)(2-0+1)}\,|2-1\rangle = \hbar\sqrt{6}\,|2-1\rangle$

$$\therefore\ |2-1\rangle = \frac{1}{\sqrt{2}}\left(|1-110\rangle + |101-1\rangle\right)$$ //

(5) $|2\ {-2}\rangle$ を求めよ。

$$J_-|2\ {-1}\rangle = (J_{1-} + J_{2-})\frac{1}{\sqrt{2}}\left(|1\ {-1}\ 1\ 0\rangle + |1\ 0\ 1\ {-1}\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left\{\underbrace{J_{1-}|1\ {-1}\rangle|1\ 0\rangle}_{=0} + \underbrace{J_{1-}|1\ 0\rangle}_{=\hbar\sqrt{1\cdot 2}|1\ {-1}\rangle}|1\ {-1}\rangle + |1\ {-1}\rangle\underbrace{J_{2-}|1\ 0\rangle}_{=\hbar\sqrt{1\cdot 2}|1\ {-1}\rangle} + \underbrace{|1\ 0\rangle J_{2-}|1\ {-1}\rangle}_{=0}\right\}$$

$$= \hbar\left(|1\ {-1}\ 1\ {-1}\rangle + |1\ {-1}\ 1\ {-1}\rangle\right)$$

$$= 2\hbar|1\ {-1}, 1\ {-1}\rangle$$

一方、$J_-|2\ {-1}\rangle = \hbar\sqrt{(2-1)(2+1+1)}\ |2\ {-2}\rangle = 2\hbar|2\ {-2}\rangle$

$\therefore\ |2\ {-2}\rangle = |1\ {-1}, 1\ {-1}\rangle$ //

(6) $|1\ 1\rangle = a|1\ 0\ 1\ 1\rangle + b|1\ 1\ 1\ 0\rangle$ とおき、$|2\ 1\rangle$ と直交するように定めよ。

なお $\langle 1\ 1|1\ 1\rangle = 1$ であり、$b > 0$ とせよ。

$$\langle 2\ 1|1\ 1\rangle = \frac{1}{\sqrt{2}}\left(\langle 1011| + \langle 1110|\right)\left(a|1011\rangle + b|1110\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\Big(a\underbrace{\langle 1011|1011\rangle}_{=1} + b\underbrace{\langle 1011|1110\rangle}_{=0} + a\underbrace{\langle 1110|1011\rangle}_{=0} + b\underbrace{\langle 1110|1110\rangle}_{=1}\Big)$$

$$= a + b = 0$$

又、$\langle 1\ 1|1\ 1\rangle = a^2 + b^2 = 1$

$2b^2 = 1$, $b > 0$ より $b = \frac{1}{\sqrt{2}}$

$\therefore\ a = -\frac{1}{\sqrt{2}}$

よって、$|1\ 1\rangle = \frac{1}{\sqrt{2}}\left(-|1011\rangle + |1110\rangle\right)$ //

(7) $|10\rangle$を求めよ.

$$J_-|11\rangle = (J_{1-} + J_{2-})\frac{1}{\sqrt{2}}\left(-|10\rangle|11\rangle + |11\rangle|10\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left\{-\underbrace{J_{1-}|10\rangle}_{=\hbar\sqrt{1\cdot 2}|1-1\rangle}|11\rangle + \underbrace{J_{1-}|11\rangle}_{=\hbar\sqrt{2\cdot 1}|10\rangle}|10\rangle - |10\rangle \underbrace{J_{2-}|11\rangle}_{=\hbar\sqrt{2\cdot 1}|10\rangle} + |11\rangle \underbrace{J_{2-}|10\rangle}_{=\hbar\sqrt{1\cdot 2}|1-1\rangle}\right\}$$

$$= \hbar\left(-|1-1\,11\rangle + |11\,1-1\rangle\right)$$

一方, $J_-|11\rangle = \hbar\sqrt{(1+1)(1-1+1)}\,|10\rangle = \hbar\sqrt{2}|10\rangle$

$$\therefore\ |10\rangle = \frac{1}{\sqrt{2}}\left(-|1-1\,11\rangle + |11\,1-1\rangle\right)$$

(8) $|1-1\rangle$を求めよ。

$$J_-|10\rangle = (J_{1-} + J_{2-})\frac{1}{\sqrt{2}}\left(-|1-1\,11\rangle + |11\,1-1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left\{-\underbrace{J_{1-}|1-1\rangle}_{=0}|11\rangle + \underbrace{J_{1-}|11\rangle}_{=\hbar\sqrt{2\cdot 1}|10\rangle}|1-1\rangle - |1-1\rangle \underbrace{J_{2-}|11\rangle}_{=\hbar\sqrt{2\cdot 1}|10\rangle} + |11\rangle \underbrace{J_{2-}|1-1\rangle}_{=0}\right\}$$

$$= \hbar\left(|10\,1-1\rangle - |1-1\,10\rangle\right)$$

一方, $J_-|10\rangle = \hbar\sqrt{(1+0)(1-0+1)}\,|1-1\rangle = \hbar\sqrt{2}|1-1\rangle$

$$\therefore\ |1-1\rangle = \frac{1}{\sqrt{2}}\left(|10\,1-1\rangle - |1-1\,10\rangle\right)$$

(9) $J_-|1-1\rangle$ を計算せよ。

$$J_-|1-1\rangle = (J_{1-} + J_{2-})\frac{1}{\sqrt{2}}\left(|10\,1-1\rangle + |1-1\,10\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(\underbrace{J_{1-}|10\rangle}_{=\hbar\sqrt{2}|1-1\rangle}|1-1\rangle - \underbrace{J_{1-}|1-1\rangle}_{=0}|10\rangle + |10\rangle\underbrace{J_{2-}|1-1\rangle}_{=0} - |1-1\rangle\underbrace{J_{2-}|10\rangle}_{=\hbar\sqrt{2}|1-1\rangle}\right)$$

$$= \hbar\left(|1-1\,1-1\rangle - |1-1\,1-1\rangle\right)$$

$$= 0$$

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(10) $\Phi_1 = (|21\rangle, |11\rangle)$, $\psi_1 = (|1011\rangle, |1110\rangle)$ として $\Phi_1 = \psi_1 M_1$ となる2行2列の行列 M_1 を求めよ。

- $|21\rangle = \frac{1}{\sqrt{2}}|1011\rangle + \frac{1}{\sqrt{2}}|1110\rangle$
- $|11\rangle = \frac{-1}{\sqrt{2}}|1011\rangle + \frac{1}{\sqrt{2}}|1110\rangle$ だから、

$$(|21\rangle, |11\rangle) = (|1011\rangle, |1110\rangle)\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$ と書ける。

$$\therefore M = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

(11) $\Phi_0 = (|20\rangle, |10\rangle, |00\rangle)$, $\psi_0 = (|1\text{-}111\rangle, |1010\rangle, |111\text{-}1\rangle)$ として、

$|00\rangle = \psi_0 \psi$ を満たす3成分の列ベクトル ψ を以下の方法で定めよう。

(i) $|20\rangle = \psi_0 \psi_{c1}$ を満たす3成分の列ベクトル ψ_{c1} を求めよ。

$|20\rangle = \frac{1}{\sqrt{6}}|1\text{-}111\rangle + \frac{2}{\sqrt{6}}|1010\rangle + \frac{1}{\sqrt{6}}|111\text{-}1\rangle$ だから、

$$|20\rangle = (|1\text{-}111\rangle, |1010\rangle, |111\text{-}1\rangle)\begin{pmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}$$

$$\therefore \psi_{c1} = \begin{pmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}$$

(ii) $|10\rangle = \psi_0 \psi_{c2}$ を満たす3成分の列ベクトル ψ_{c2} を求めよ。

$|10\rangle = \frac{-1}{\sqrt{2}}|1\text{-}111\rangle + \frac{1}{\sqrt{2}}|111\text{-}1\rangle$ だから、

$$|10\rangle = (|1\text{-}111\rangle, |1010\rangle, |111\text{-}1\rangle)\begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$$

$$\therefore \psi_{c2} = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$$

(iii) 2行2列行列 $\psi_c = (\psi_{c1}, \psi_{c2})$ を用いて $P = \psi_c \psi_c^\dagger$ を計算せよ。

$$P = \psi_c \psi_c^\dagger = (\psi_{c1}, \psi_{c2})\begin{pmatrix} \psi_{c1}^\dagger \\ \psi_{c2}^\dagger \end{pmatrix}$$

$$= (\psi_{c1}\psi_{c1}^\dagger + \psi_{c2}\psi_{c2}^\dagger)$$

$$P = \begin{pmatrix} 1/6 & 2/6 & 1/6 \\ 2/6 & 4/6 & 2/6 \\ 1/6 & 2/6 & 1/6 \end{pmatrix} + \begin{pmatrix} 1/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{pmatrix} = \underline{\begin{pmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{pmatrix}}_{\,/\!/}$$

(iv) 3成分の列ベクトル $\phi = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ を用いて $\psi' = (E_3 - P)\phi$ を求めよ。

E_3 は3次元単位行列である。

$$\psi' = (E_3 - P)\phi$$

$$= \begin{pmatrix} 1/3 & -1/3 & 1/3 \\ -1/3 & 1/3 & -1/3 \\ 1/3 & -1/3 & 1/3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \underline{\begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \end{pmatrix}}_{\,/\!/}$$

(v) $\psi^\dagger_{c1}\psi' = \psi_{c2}^\dagger \psi' = 0$ を示せ。

$$\psi^\dagger_{c1}\psi' = \begin{pmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \end{pmatrix} \begin{pmatrix} 1/3 \\ -1/3 \\ 1/3 \end{pmatrix}$$

$$= \frac{1}{3\sqrt{6}}\left(1 - 2 + 1\right) = \underline{0}_{\,/\!/}$$

$$\psi_{c2}^\dagger \psi' = \begin{pmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/3 \\ -1/3 \\ 1/3 \end{pmatrix}$$

$$= \frac{1}{3\sqrt{2}}\left(-1 + 0 + 1\right) = \underline{0}_{\,/\!/}$$

(vi) $\psi = \dfrac{\psi'}{\|\psi'\|}$ として $|00\rangle = \psi_0 \psi$ であることを確認しよう。

(a) $J_z|00\rangle$ を計算せよ。

$$\|\psi'\| = \sqrt{\left(\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = \sqrt{\frac{1}{9} + \frac{1}{9} + \frac{1}{9}} = \frac{1}{\sqrt{3}}$$

$\therefore\ \psi = \begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$ であり $|00\rangle = \psi_0 \psi = \left(|1{-1}11\rangle, |1010\rangle, |111{-1}\rangle\right) \begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$

$$J_z|00\rangle = (J_{1z} + J_{2z}) \frac{1}{\sqrt{3}} \left(|1{-1}11\rangle - |1010\rangle + |111{-1}\rangle \right)$$

$$= \frac{1}{\sqrt{3}} \Big(\underbrace{J_{1z}|1{-1}\rangle}_{= -|1{-1}\rangle}|11\rangle - \underbrace{J_{1z}|10\rangle}_{=0}|10\rangle + \underbrace{J_{1z}|11\rangle}_{=|11\rangle}|1{-1}\rangle + |1{-1}\rangle \underbrace{J_{2z}|11\rangle}_{=|11\rangle} - |10\rangle \underbrace{J_{2z}|10\rangle}_{=0}$$

$$+ |11\rangle \underbrace{J_{2z}|1{-1}\rangle}_{= -|1{-1}\rangle} \Big)$$

$$J_z|00\rangle = \frac{1}{\sqrt{3}}\left(-|1-111\rangle + |111-1\rangle + |1-111\rangle - |111-1\rangle\right) = 0$$

(b) $J_+|00\rangle$ を計算せよ。

$$J_+|00\rangle = (J_{1+} + J_{2+})\frac{1}{\sqrt{3}}\left(|1-111\rangle - |1010\rangle + |111-1\rangle\right)$$

$$= \frac{1}{\sqrt{3}}\Big(\underbrace{J_{1+}|1-1\rangle}_{=\hbar\sqrt{2\cdot1}|10\rangle}|11\rangle - \underbrace{J_{1+}|10\rangle}_{=\hbar\sqrt{1\cdot2}|11\rangle}|10\rangle + \underbrace{J_{1+}|11\rangle}_{=0}|1-1\rangle + |1-1\rangle\underbrace{J_{2+}|11\rangle}_{=0} - |10\rangle\underbrace{J_{2+}|10\rangle}_{=\hbar\sqrt{1\cdot2}|11\rangle} + |11\rangle\underbrace{J_{2+}|1-1\rangle}_{=\hbar\sqrt{2\cdot1}|10\rangle}\Big)$$

$$= \hbar\sqrt{\frac{2}{3}}\left(|1011\rangle - |1110\rangle - |1011\rangle + |1110\rangle\right) = 0$$

(c) $J_-|00\rangle$ を計算せよ。

$$J_-|00\rangle = (J_{1-} + J_{2-})\frac{1}{\sqrt{3}}\left(|1-111\rangle - |1010\rangle + |111-1\rangle\right)$$

$$= \frac{1}{\sqrt{3}}\Big(\underbrace{J_{1-}|1-1\rangle}_{=0}|11\rangle - \underbrace{J_{1-}|10\rangle}_{=\hbar\sqrt{1\cdot2}|1-1\rangle}|10\rangle + \underbrace{J_{1-}|11\rangle}_{=\hbar\sqrt{2\cdot1}|10\rangle}|1-1\rangle + |1-1\rangle\underbrace{J_{2-}|11\rangle}_{=\hbar\sqrt{2\cdot1}|10\rangle} - |10\rangle\underbrace{J_{2-}|10\rangle}_{=\hbar\sqrt{1\cdot2}|1-1\rangle} + |11\rangle\underbrace{J_{2-}|1-1\rangle}_{=0}\Big)$$

$$= \hbar\sqrt{\frac{2}{3}}\left(-|1-110\rangle + |101-1\rangle + |1-110\rangle - |101-1\rangle\right) = 0$$

(12) $\Psi_0 = \varphi_0 M_0$ となる3行3列の行列 M_0 を求めよ。

- $|20\rangle = \frac{1}{\sqrt{6}}|1-111\rangle + \frac{2}{\sqrt{6}}|1010\rangle + \frac{1}{\sqrt{6}}|111-1\rangle$
- $|10\rangle = -\frac{1}{\sqrt{2}}|1-111\rangle + \frac{1}{\sqrt{2}}|111-1\rangle$
- $|00\rangle = \frac{1}{\sqrt{3}}|1-111\rangle - \frac{1}{\sqrt{3}}|1010\rangle + \frac{1}{\sqrt{3}}|111-1\rangle$

だから、

$$(|20\rangle, |10\rangle, |00\rangle) = (|1-111\rangle, |1010\rangle, |111-1\rangle)\begin{pmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix}$$

$$\therefore M_0 = \begin{pmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix}$$

(13) ゼロでない $\langle 1\, m_1\, 1\, m_2 | j\, m \rangle$ を全て書き出せ。

$j=2,\ m=2$ のとき、 $|22\rangle = |1111\rangle$

$$\Rightarrow \quad \langle 1111 | 22 \rangle = 1$$

$j=2,\ m=1$ のとき $|21\rangle = \frac{1}{\sqrt{2}}\left(|1011\rangle + |1110\rangle\right)$

$$\Rightarrow \quad \begin{aligned} \langle 1011 | 21 \rangle &= \frac{1}{\sqrt{2}} \\ \langle 1110 | 21 \rangle &= \frac{1}{\sqrt{2}} \end{aligned}$$

$j=2,\ m=0$ のとき、 $|20\rangle = \frac{1}{\sqrt{6}}\left(|1{-1}11\rangle + 2|1010\rangle + |111{-1}\rangle\right)$

$$\Rightarrow \quad \begin{aligned} \langle 1{-1}11 | 20 \rangle &= \frac{1}{\sqrt{6}} \\ \langle 1010 | 20 \rangle &= \frac{2}{\sqrt{6}} \\ \langle 111{-1} | 20 \rangle &= \frac{1}{\sqrt{6}} \end{aligned}$$

$j=2,\ m=-1$ のとき、 $|2{-1}\rangle = \frac{1}{\sqrt{2}}\left(|1{-1}10\rangle + |101{-1}\rangle\right)$

$$\Rightarrow \quad \begin{aligned} \langle 1{-1}10 | 2{-1} \rangle &= \frac{1}{\sqrt{2}} \\ \langle 101{-1} | 2{-1} \rangle &= \frac{1}{\sqrt{2}} \end{aligned}$$

$j=2,\ m=-2$ のとき、 $|2{-2}\rangle = |1{-1}1{-1}\rangle$

$$\Rightarrow \quad \langle 1{-1}1{-1} | 2{-2} \rangle = 1$$

$j=1,\ m=1$ のとき、 $|11\rangle = \frac{1}{\sqrt{2}}\left(-|1011\rangle + |1110\rangle\right)$

$$\Rightarrow \quad \begin{aligned} \langle 1011 | 11 \rangle &= -\frac{1}{\sqrt{2}} \\ \langle 1110 | 11 \rangle &= \frac{1}{\sqrt{2}} \end{aligned}$$

$j=1,\ m=0$ のとき、 $|10\rangle = \frac{1}{\sqrt{2}}\left(-|1{-1}11\rangle + |111{-1}\rangle\right)$

$$\Rightarrow \quad \begin{aligned} \langle 1{-1}11 | 10 \rangle &= -\frac{1}{\sqrt{2}} \\ \langle 111{-1} | 10 \rangle &= \frac{1}{\sqrt{2}} \end{aligned}$$

$j=1,\ m=-1$ のとき $|1{-1}\rangle = \frac{1}{\sqrt{2}}\left(|101{-1}\rangle - |1{-1}10\rangle\right)$

$$\Rightarrow \quad \begin{aligned} \langle 101{-1} | 1{-1} \rangle &= \frac{1}{\sqrt{2}} \\ \langle 1{-1}10 | 1{-1} \rangle &= -\frac{1}{\sqrt{2}} \end{aligned}$$

(裏に続く)

$j=0$, $m=0$ のとき、 $|00\rangle = \frac{1}{\sqrt{3}}\left(|1-111\rangle - |1010\rangle + |11-1\rangle\right)$

$$\Rightarrow \quad \langle 1-111|00\rangle = \frac{1}{\sqrt{2}}$$

$$\langle 1010|00\rangle = \frac{-1}{\sqrt{3}}$$

$$\langle 111-1|00\rangle = \frac{1}{\sqrt{3}}$$

従って、

- $\langle 1111|22\rangle = 1$
- $\langle 1011|21\rangle = \frac{1}{\sqrt{2}}$
- $\langle 1110|21\rangle = \frac{1}{\sqrt{2}}$
- $\langle 1-111|20\rangle = \frac{1}{\sqrt{6}}$
- $\langle 1010|20\rangle = \frac{2}{\sqrt{6}}$
- $\langle 111-1|20\rangle = \frac{1}{\sqrt{6}}$
- $\langle 1-110|2-1\rangle = \frac{1}{\sqrt{2}}$
- $\langle 101-1|2-1\rangle = \frac{1}{\sqrt{2}}$
- $\langle 1-11-1|2-2\rangle = 1$
- $\langle 1011|11\rangle = -\frac{1}{\sqrt{2}}$
- $\langle 1110|11\rangle = \frac{1}{\sqrt{2}}$
- $\langle 1-111|10\rangle = -\frac{1}{\sqrt{2}}$
- $\langle 111-1|10\rangle = \frac{1}{\sqrt{2}}$
- $\langle 101-1|1-1\rangle = \frac{1}{\sqrt{2}}$
- $\langle 1-110|1-1\rangle = -\frac{1}{\sqrt{2}}$
- $\langle 1-111|00\rangle = \frac{1}{\sqrt{3}}$
- $\langle 1010|00\rangle = -\frac{1}{\sqrt{3}}$
- $\langle 111-1|00\rangle = \frac{1}{\sqrt{3}}$