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荻野孝浩

量子力学3

①

$\vec{r} \mapsto \vec{r}+\vec{a}$ $\quad \psi'(\vec{r}) = \psi(\vec{r}-\vec{a})$

$\vec{a} \to \delta\vec{a}$ とかくと

$\delta\psi = -\delta\vec{a}\cdot\nabla\psi$

$U(\delta\vec{a}) \equiv e^{-i\delta\vec{a}\cdot\vec{p}}$: 無限小変換

$U = e^{i\lambda G/\hbar}$ $\quad G^\dagger = G$

G : 変換の母関数

$U^\dagger = U^{-1}$ $\quad UU^\dagger = 1$

unitary 変換

物理量の変換 $\quad O = O^\dagger$

$\langle\psi|O|\psi\rangle = \langle\psi'|O'|\psi'\rangle = \langle\psi|U^\dagger O' U|\psi\rangle$

$\Rightarrow O = U^\dagger O' U \quad \Rightarrow O' = U O U^\dagger$

特に $U = e^{i\delta\lambda G/\hbar}$ に対しては

$(= 1+i\delta\lambda G/\hbar)$

$O' = (1-i\delta\lambda G/\hbar)O(1+i\delta\lambda G/\hbar) = O + \frac{i\delta\lambda}{\hbar}[G, O]$

$\delta O = i\frac{\delta\lambda}{\hbar}[G, O]$

無限小の並進 $\quad U = 1 - i\delta\vec{a}\cdot\vec{p}/\hbar$

$\delta\vec{p} = -i[\delta\vec{a}\cdot\vec{p}, \vec{p}]/\hbar$

運動量は並進で不変

$\delta p_i = -\delta a_j[p_j, p_i]/\hbar = 0$

$\delta\vec{r} = -i[\delta\vec{a}\cdot\vec{p}, \vec{r}]/\hbar$

$\delta r_i = -i\delta a_j[p_j, r_i]/\hbar = -\delta a_i \quad (\because [r_i, p_j] = i\hbar\delta_{ij})$

$\delta\vec{r} = \vec{r}' - \vec{r} = -\delta\vec{a}$

$\langle\psi'|\vec{r}'|\psi'\rangle = \langle\psi|\vec{r}|\psi\rangle = \int d^3r\,|\psi(\vec{r})|^2\vec{r}$

$\langle\psi(\vec{r}-\vec{a})|\vec{r}-\vec{a}|\psi(\vec{r}-\vec{a})\rangle = \int d^3r\,|\psi(\vec{r}-\vec{a})|^2(\vec{r}-\vec{a})$

特に時間に陽に依存しない H が不変 つまり $\delta H = i\delta\lambda[G, H]/\hbar = 0$
なら

$$i\hbar\partial_t|\psi\rangle = H|\psi\rangle \qquad |\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle$$

$$\langle G\rangle = \langle\psi(t)|G|\psi(t)\rangle = \langle\psi(0)|e^{iHt/\hbar}Ge^{-iHt/\hbar}|\psi(0)\rangle$$

$$\frac{d}{dt}\langle G\rangle = \langle\psi(0)|e^{iHt/\hbar}\underset{\overset{\shortparallel}{0}}{[H,G]}e^{-iHt/\hbar}|\psi(0)\rangle\left(\frac{i}{\hbar}\right)$$

$$= 0$$ 保存量となる.

ex) $H = \dfrac{\vec{p}^2}{2m}$ 自由粒子

$$= \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2)$$

$[H, \vec{p}] = 0$ $U = e^{-i\delta\vec{a}\cdot\vec{p}/\hbar}$ で不変

(並進操作で不変)

運動量 $\vec{p}$は保存量となる.

空間回転 $\vec{r} \mapsto \vec{r}' = R\vec{r}$

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad R = \begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix}$$

線型変換

例

$$R = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ z軸まわり θ 回転

回転ベクトルの長さ不変 $|\vec{r}|^2 = \tilde{\vec{r}}\vec{r} = (x, y, z)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x^2 + y^2 + z^2$

$$|\vec{r}'|^2 = \tilde{\vec{r}}'\vec{r}' = (\widetilde{R\vec{r}})R\vec{r} = \tilde{\vec{r}}\tilde{R}R\vec{r}$$

$$\therefore \quad \tilde{R}R = E_3 = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$$

$$\det\tilde{R}R = \det\tilde{R}\det R = (\det R)^2 = \det E_3 = 1$$

$$\det R = \pm 1$$

$\det R = 1$のときを考える

($R = E_3$に連続変形できるものを考える.)

無限小回転 δR

$$R = E_3 + \delta R = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix} + \Big(\ \delta R\ \Big)$$

$$\tilde{R} = 1 + \widetilde{\delta R} \qquad \tilde{R}R = 1$$
$$= (1+\widetilde{\delta R})(1+\delta R)$$
$$= 1 + \widetilde{\delta R} + \delta R$$

$\widetilde{\delta R} + \delta R = 0$ 反対称

$$(\delta R)_{ij} = -(\delta R)_{ji}$$

$(\delta R)_{ij} = \varepsilon_{ijk}\,\delta\omega_k$ と書く

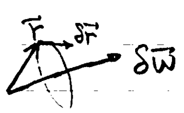
$\delta\omega_k$：無限小の実数

$$\vec{r} \longmapsto \vec{r}\,' = R\vec{r} = (1+\delta R)\vec{r}$$

$$\delta\vec{r} = \vec{r}\,' - \vec{r} = \delta R\vec{r}$$

$$(\delta\vec{r})_i = \delta R_{ij} r_j = -\varepsilon_{ijk}\,\delta\omega_k\, r_j = \varepsilon_{ikj}\,\delta\omega_k\, r_j$$

$$\delta\vec{r} = \delta\vec{\omega} \times \vec{r}$$



$\delta\vec{\omega}$方向に右ネジが進むときネジの回転する方向

$$\delta\psi = \psi'(\vec{r}) - \psi(\vec{r})$$
$$= \psi(\tilde{R}\vec{r}) - \psi(\vec{r})$$
$$= \psi((1-\delta R)\vec{r}) - \psi(\vec{r})$$
$$= \psi(\vec{r} - \delta\vec{r}) - \psi(\vec{r})$$
$$= -\delta\vec{r}\cdot\nabla\psi = -i\delta\vec{r}\cdot\vec{p}\,\psi/\hbar$$
$$= -i(\delta\vec{\omega}\times\vec{r})\cdot\vec{p}\,\psi/\hbar$$
$$= -i(\vec{r}\times\vec{p})\cdot\delta\vec{\omega}\,\psi/\hbar$$
$$= -i\,\delta\vec{\omega}\cdot\underbrace{(\vec{r}\times\vec{p})}_{\vec{L}}\psi/\hbar$$

$$\psi'(\vec{r}\,') = \psi'(R\vec{r}) = \psi(\vec{r})$$
$$\psi'(\vec{r}) = \psi(R^{-1}\vec{r})$$
$$= \psi(\tilde{R}\vec{r})$$

$$\vec{L} = \vec{r}\times\vec{p}$$

$$\delta\psi = -i\delta\vec{\omega}\cdot\vec{L}\,\psi/\hbar$$

cf. 並進 $\delta\psi = -i\delta\vec{a}\cdot\vec{p}\,\psi/\hbar$

一般に $U = e^{-i\vec{\omega}\cdot\vec{L}/\hbar}$

無限小変換 $\delta U = 1 - i\delta\vec{\omega}\cdot\vec{L}/\hbar = e^{-i\delta\vec{\omega}\cdot\vec{L}/\hbar}$

$$\delta O = [-i\delta\vec{\omega}\cdot\vec{L}/\hbar,\ O]$$

例 $\delta r_i = -\left(\frac{i}{\hbar}\right)[\delta\vec{\omega}\cdot\vec{L}, r_i] = -\frac{i}{\hbar}\sum_j \delta\omega_j [L_j, r_i]$

$= -\frac{i}{\hbar}\sum_j \delta\omega_j [\varepsilon_{jab} r_a p_b, r_i]$

$= -\frac{i}{\hbar}\sum_j \delta\omega_j\, \varepsilon_{jab} r_a [p_b, r_i]$

$= -\sum_j \delta\omega_j\, \varepsilon_{jab} r_a \delta_{bi}$

$= -\sum_j \delta\omega_j\, \varepsilon_{jai} r_a$

$= -\sum_j \varepsilon_{ija}\, \delta\omega_j r_a = -(\delta\vec{\omega}\times\vec{r})_i$

$$\delta\vec{r} = -\delta\vec{\omega}\times\vec{r}$$

$\delta\vec{p} = [-i\delta\vec{\omega}\cdot\vec{L}, \vec{p}]/\hbar$

$\delta p_i = -i\,\delta\omega_j [L_j, p_i]/\hbar$ （アインシュタインの記法を用いる（j））

$= -i\,\delta\omega_j [\varepsilon_{jab} r_a p_b, p_i]/\hbar$

$= -i\,\delta\omega_j\, \varepsilon_{jab} \underbrace{[r_a p_b, p_i]}_{=[r_a, p_i] p_b = i\hbar\delta_{ai} p_b}/\hbar$

$= \delta\omega_j\, \varepsilon_{jab}\, \delta_{ai}\, p_b$

$= \delta\omega_j\, \varepsilon_{jib}\, p_b$

$= -\varepsilon_{ijb}\, \delta\omega_j\, p_b$

$$\delta\vec{p} = -\delta\vec{\omega}\times\vec{p}$$

一般に $\delta V_i = -\delta\vec{\omega}\times\vec{V}$ と変換するものをベクトルとして変換するという。

$\vec{r}, \vec{p}$ はベクトル

$\vec{V}^2 = \vec{V}\cdot\vec{V}$ $\quad \delta V^2 = [-i\delta\vec{\omega}\cdot\vec{L}, \vec{V}\cdot\vec{V}]/\hbar$

$= [-i\delta\vec{\omega}\cdot\vec{L}/\hbar, \vec{V}]\cdot\vec{V} + \vec{V}\cdot[-i\delta\vec{\omega}\cdot\vec{L}/\hbar, \vec{V}] = 0$

$-(\delta\vec{\omega}\times\vec{V})\cdot\vec{V} = 0$

スカラー

$-\vec{V}\cdot(\delta\vec{\omega}\times\vec{V}) = 0$

$H = \vec{p}^2/2m$ $\quad \delta H = 0 \quad \Rightarrow \vec{L}$ が保存

回転に対してスカラーとして変換（不変）

$H = \frac{\vec{p}^2}{2m} + \frac{1}{2}m\omega^2\vec{r}^2$ 調和振動子 $\rightarrow L$ は保存