A

spherical harmonics (conti.)

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times \frac{\hbar}{i} \vec{\nabla}$$

$$= \hbar \begin{pmatrix} i\sin\phi \\ -i\cos\phi \\ 0 \end{pmatrix} \partial_\theta + \hbar \begin{pmatrix} i\cos\phi\cot\theta \\ i\sin\phi\cot\theta \\ -i \end{pmatrix} \partial_\phi = \begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix}$$

だから、

$$\begin{cases} L_+ = L_x + iL_y = \hbar e^{i\phi} (\partial_\theta + i\cot\theta\,\partial_\phi) \\ L_- = L_x - iL_y = \hbar e^{-i\phi} (-\partial_\theta + i\cot\theta\,\partial_\phi) \\ L_z = -i\hbar\,\partial_\phi = \frac{\hbar}{i}\partial_\phi \end{cases}$$

となる。

$Y_{lm}(\theta,\phi) = \Theta(\theta)\,\Phi(\phi)$: spherical harmonics として

$$\begin{cases} L^2 Y_{lm} = \hbar^2 l(l+1) Y_{lm} & \text{—— ①} \\ L_z Y_{lm} = \hbar m Y_{lm} & \text{—— ②} \end{cases}$$

② は $\frac{\hbar}{i}\partial_\phi \Phi = \hbar m \Phi \quad \rightarrow \Phi \propto e^{im\phi}$

さらに $\Phi(\phi + 2\pi) = \Phi(\phi)$ なので $e^{i2\pi m} = 1 \rightarrow m$: integers

規格化して $\int_0^{2\pi} d\Phi |\Phi|^2 = 1 \quad \rightarrow \underline{\Phi(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}}$ となる。

$$L_+[e^{im\phi} f(\theta)] = \hbar e^{i\phi} (e^{im\phi} f'(\theta) + i\cot\theta (im) e^{im\phi} f(\theta))$$

$$= \hbar e^{i(m+1)\phi} (f'(\theta) - m\cot\theta f(\theta))$$

$$\left(\begin{aligned} &\text{ここで} \quad \frac{d}{d\theta} = \frac{d(\cos\theta)}{d\theta} \frac{d}{d(\cos\theta)} = -\sin\theta \frac{d}{d(\cos\theta)} \\ &\frac{d\sin\theta}{d\cos\theta} = \frac{d(1-\cos^2\theta)^{\frac{1}{2}}}{d\cos\theta} = \frac{1}{2}(1-\cos^2\theta)^{-\frac{1}{2}} \cdot (-2\cos\theta) = -\frac{\cos\theta}{\sin\theta} = -\cot\theta \end{aligned} \right)$$

を用いれば

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$$= \hbar e^{i(m+1)\phi}\left(-\sin\theta \frac{df}{d\cos\theta} + mf(\theta)\frac{d\sin\theta}{d\cos\theta}\right)$$

(ここで $\frac{d}{d\cos\theta}(\sin^{-m}\theta f)$

$$= \sin^{-m-1}\theta \frac{d\sin\theta}{d\cos\theta} f + \sin^{-m}\theta \frac{df}{d\cos\theta}$$ なので、)

$$= -\hbar e^{i(m+1)\phi} \sin^{m+1}\theta \frac{d}{d\cos\theta}(\sin^{-m}\theta f(\theta))$$

なので、$L_+^2[e^{im\phi}f(\theta)] = L_+(-\hbar e^{i(m+1)\phi}\sin^{m+1}\theta \frac{d}{d\cos\theta}(\sin^{-m}\theta f(\theta)))$

$$= (-\hbar)^2 e^{i(m+2)\phi}\sin^{m+2}\theta \frac{d}{d\cos\theta}(\cancel{\sin^{-(m+1)}\theta}\,\cancel{\sin^{m+1}\theta}\frac{d}{d\cos\theta}(\sin^{-m}\theta f))$$

$$= (-\hbar)^2 e^{i(m+2)\phi}\sin^{m+2}\theta\left(\frac{d}{d\cos\theta}\right)^2(\sin^{-m}\theta f)$$

よって $L_+^k[e^{im\phi}f(\theta)] = (-\hbar)^k e^{i(m+k)\phi}\sin^{m+k}\theta\left(\frac{d}{d\cos\theta}\right)^k(\sin^{-m}\theta f)$

同様にして

$L_-[e^{im\phi}f(\theta)] = \hbar e^{i(m-1)\phi}\sin^{-(m-1)}\theta\left(\frac{d}{d\cos\theta}\right)^1(\sin^{m}\theta f(\theta))$ より、

$L_-^k[e^{im\phi}f(\theta)] = \hbar^k e^{i(m-k)\phi}\sin^{-(m-k)}\theta\left(\frac{d}{d\cos\theta}\right)^k(\sin^{m}\theta f(\theta))$

とわかる。

これらを用いて

$Y_{\ell\ell}$ は $L_+Y_{\ell\ell} = 0 = L_+[\frac{1}{\sqrt{2\pi}}e^{i\ell\phi}\Theta_{\ell\ell}(\theta)]$
$(m=\ell)$

$$= -\hbar e^{i(\ell+1)\phi}\sin^{\ell+1}(\theta)\frac{d}{d\cos\theta}\left[\sin^{-\ell}\theta\,\Theta_{\ell\ell}(\theta)\frac{1}{\sqrt{2\pi}}\right] = 0$$

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ここで $\frac{d}{d\cos\theta}\left(-\sin^{-\ell}\theta\, \Theta_{\ell\ell}(\theta)\right)$

$$= \ell \sin^{-\ell-1}\theta(-\cot\theta)\Theta - \underbrace{\sin^{-\ell}\theta}_{\sin^{-\ell-1}\theta\cdot\sin\theta} \frac{d\Theta}{d\cos\theta} \qquad (\because \frac{d\sin\theta}{d\cos\theta} = -\cot\theta)$$

$$= \sin^{-\ell-1}\theta\left(-\ell\cot\theta\,\Theta + \frac{d}{d\theta}\Theta\right) \qquad (\because \frac{d}{d\theta} = -\sin\theta\frac{d}{d\cos\theta})$$

元に戻して

$$= -\hbar e^{i(\ell+1)\phi}\left(-\ell\cot\theta\,\Theta + \frac{d\Theta}{d\theta}\right) = 0 \qquad -*$$

$$\therefore \frac{d\Theta}{d\theta} = \ell\cot\Theta$$

($\Theta = c\sin^{\ell}\theta$ とすれば

$$\frac{d\Theta}{d\theta} = c\ell\sin^{\ell-1}\theta\cos\theta$$
$$= c\ell\sin^{\ell}\theta\cot\theta$$
$$= \ell\cot\theta\,\Theta$$ で成立している)

規格化で、

$$\int_0^{\pi} d\theta \int_0^{2\pi} d\phi \quad * \quad = 1$$

$$\int_0^{\pi} d\theta\, |\Theta(\theta)|^2 \int_0^{2\pi} d\phi = 1$$

$$= |c|^2 \int_0^{\pi} d\theta \sin^{2\ell+1}\theta = 1$$ Beta fc. でとけば、

$$c = (-)^{\ell}\sqrt{\frac{(2\ell+1)!}{2}}\frac{1}{2^{\ell}\ell!}$$

$$Y_{\ell\ell} = (-)^{\ell}\frac{1}{\sqrt{2\pi}}\sqrt{\frac{(2\ell+1)!}{2}}\frac{1}{2^{\ell}\ell!}e^{i\ell\phi}\sin^{\ell}\theta$$

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ここで
$$|jm\rangle = \hbar^{-k}\left[\frac{(j+m)!}{(j+m+k)!}\ \frac{(j-m-k)!}{(j-m)!}\right]^{\frac{1}{2}} J_-^{k}\ |j\ \ m+k\rangle$$

で、 $m=0$, $j=l$, $k=l$ とすれば、

$$|l\,0\rangle = \hbar^{-l}\left[\frac{\cancel{l!}}{(2l)!}\ \cdot\ \frac{0!}{\cancel{l!}}\right]^{\frac{1}{2}} L_-^{l}\ |l\ l\rangle$$ になる。

$$Y_{l0} = \hbar^{-l}\frac{1}{\cancel{\sqrt{(2l)!}}} L_-^{l}\ \underbrace{Y_{ll}}_{=\,(-)^{l}\frac{1}{\sqrt{2\pi}}\cancel{\sqrt{\frac{(2l+1)!}{2}}}\sqrt{\frac{2l+1}{2}}\frac{1}{2^{l}l!}e^{il\phi}\sin^{l}\theta}$$

$$= \cancel{\hbar^{-l}}(-1)^{l}\frac{1}{\sqrt{2\pi}}\sqrt{\frac{2l+1}{2}}\frac{1}{2^{l}l!}\cancel{\hbar^{l}}\left(\frac{d}{d\cos\theta}\right)^{l}(\sin^{l}\theta\,\sin^{l}\theta)$$

$$Y_{l0} = (-1)^{l}\frac{1}{\sqrt{2\pi}}\sqrt{\frac{2l+1}{2}}\frac{1}{2^{l}l!}\left(\frac{d}{d\cos\theta}\right)^{l}\underbrace{\sin^{2l}\theta}_{=(1-\cos^{2}\theta)^{l}=(\cos^{2}\theta-1)^{l}(-1)^{l}}$$

(ここで $P_l(t) \equiv \frac{1}{2^{l}l!}\left(\frac{d}{dt}\right)^{l}(t^{2}-1)^{l}$: Legendre polynomial (ルジャンドル多項式)

と定義すれば)

$$Y_{l0}(\theta,\phi) = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{2l+1}{2}}\ P_l(\cos\theta)$$
$$= \sqrt{\frac{2l+1}{4\pi}}\ P_l(\cos\theta)$$ となる。

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$$Y_{\ell m}^{m\geq 0} = \hbar^{-k}\left[\frac{\ell!}{(\ell+m)!}\ \frac{(\ell-m)!}{\ell!}\right]^{\frac{1}{2}} L_+^{m}\, Y_{\ell 0}$$

ここで、

$$|j\ m+k\rangle = \hbar^{-m}\left[\frac{(j+m)!\,(j-m-k)!}{(j+m+k)!\,(j-m)!}\right]^{\frac{1}{2}} J_+\ |j\,m\rangle$$

で、 $j=\ell,\ m=0,\ k=m$ とすれば

$$Y_{\ell m}^{m\geq 0} = (-)^{m}\, e^{im\phi}\sqrt{\frac{2\ell+1}{4\pi}\ \frac{(\ell-m)!}{(\ell+m)!}}\ \sin^{m}\theta\left(\frac{d}{d\cos\theta}\right)^{m} P_\ell(\cos\theta) \quad //$$

一方で

$$Y_{\ell m}^{\substack{m=-m' \\ m'>0}}(+) \propto L_-^{m'}\, Y_{\ell\ell}$$ なので、

$$Y_{\ell m}^{m'>0}(+)$$

$$= Y_{\ell m'} = \sqrt{\frac{2\ell+1}{4\pi}\ \frac{(\ell-m)!}{(\ell+m)!}}\cdot e^{im\phi}\ \frac{d}{d\cos\theta}\ P_\ell(\cos\theta)$$

$$= (-1)^{|m|}\ Y_{\ell|m|}$$ である。