量子力学3 まとめレポート 5/12分

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☆ Quantization of angular momentum

$\vec{L} = \vec{r} \times \vec{p}$: any value (classical)

$L_i = \varepsilon_{ijk} r_j P_k \qquad i = x, y, z.$

$[L_i, L_j] = i\hbar \varepsilon_{ijk} L_k. \quad , \quad [r_i, P_j] = i\hbar \delta_{ij}$

Generically , $J_i \quad i = x, y, z.$

$[J_i, J_j] = i\hbar \varepsilon_{ijk} J_k \qquad J_i^\dagger = J_i$

- What is J_i ?

$\vec{J}^2 \equiv J_x^2 + J_y^2 + J_z^2 \qquad [\vec{J}^2, J_i] = 0$

$L_i^\dagger = (\varepsilon_{ijk} r_j P_k)^\dagger = \varepsilon_{ijk} P_k^\dagger r_j^\dagger = \varepsilon_{ijk} P_k r_j = \varepsilon_{ijk} r_j P_k = L_i.$

$J^2 |jm\rangle = \hbar^2 j(j+1) |j.m\rangle$ 同時固有状態

$J_z |jm\rangle = \hbar m |jm\rangle$ Simultaneous eigenstate

$m, j(j+1) \in \mathbb{R} \qquad [J_i, J_j] \neq 0 \quad (i \neq j)$

$|jm\rangle$: orthogonal and normalized $\qquad \langle jm | j'm' \rangle = \delta_{jj'} \delta_{mm'}$

$\sum_{jm} |jm\rangle\langle jm| = 1$: completeness

$\langle jm | J_z | jm \rangle = \hbar m \qquad J_z \sim \hbar m$: discretize
→ quantization

$J_\pm = J_x \pm i J_y$ とした.

— ~ formula ~ —

$$[\vec{J}^2, J_\pm] = 0 \quad ①.$$

$$[J_z, J_\pm] = \pm\hbar J_\pm \quad ②$$

$$[J_+, J_-] = 2\hbar J_z \quad ③$$

$$\vec{J}^1\cdot\vec{J}^2 = \frac{1}{2}(J_+^1 J_-^2 + J_-^1 J_+^2) + J_z^1 J_z^2 \quad ④$$

$$J_+ J_- = J^2 - J_z(J_z - \hbar) \quad ⑤$$

$$J_- J_+ = J^2 - J_z(J_z + \hbar) \quad ⑥.$$

- $\langle jm | ② | jm' \rangle$

$$\langle jm | [J_z, J_\pm] | jm' \rangle = \pm\hbar \langle jm | J_\pm | jm' \rangle$$

$$= \underbrace{\langle jm | J_z}_{=\hbar m \langle jm|} J_\pm - J_\pm \underbrace{J_z | jm' \rangle}_{\hbar m' | jm' \rangle}$$

$$(\because \langle jm | J_z = (J_z^\dagger | jm\rangle)^\dagger = (J_z | jm\rangle)^\dagger = (\hbar m | jm\rangle)^\dagger = \hbar m \langle jm|)$$

$$= \hbar(m-m') \langle jm | J_\pm | jm' \rangle$$

$$\longrightarrow \hbar(m-m') \langle jm | J_\pm | jm' \rangle = \pm\hbar \langle jm | J_\pm | jm' \rangle$$

$$\hbar(m - m' \mp 1) \langle jm | J_\pm | jm' \rangle = 0.$$

$\langle jm | J_\pm | jm' \rangle \neq 0$ only if $m - m' \mp 1 = 0 \leftrightarrow m = m' \pm 1$

$$\begin{cases} \langle jm | J_+ | jm' \rangle = 0 & \text{if } m - m' - 1 \neq 0 \\ \langle jm | J_- | jm' \rangle = 0 & \text{if } m - m' + 1 \neq 0 \end{cases}$$

∘ ② $|jm\rangle$ $\qquad [J_z, J_\pm] = J_z J_\pm - J_\pm J_z = \pm\hbar J_\pm$

$$J_z (J_\pm |jm\rangle) - \underbrace{J_\pm J_z |jm\rangle}_{= J_\pm \hbar m |jm\rangle = \hbar m J_\pm |jm\rangle} = \pm\hbar (J_\pm |jm\rangle)$$

$\longrightarrow \quad J_z (J_\pm |jm\rangle) = \hbar (m \pm 1)(J_\pm |jm\rangle)$

$\longrightarrow \quad J_\pm |jm\rangle = C_\pm |j\, m\pm1\rangle \qquad C_\pm \in \mathbb{C}$

$\left(\Leftrightarrow \begin{cases} J_+ : \text{上昇演算子} & \text{ascending operator} \\ J_- : \text{下降演算子} & \text{descending operator} \end{cases} \right)$.

$$|j\, m\pm1\rangle = C_\pm^{-1} J_\pm |jm\rangle$$

require

$$1 = \langle j\, m\pm1 | j\, m\pm1\rangle = |C_\pm|^{-2} \underbrace{(J_\pm|jm\rangle)^\dagger}_{= \langle jm| J_\pm^\dagger = \langle jm| J_\mp} J_\pm |jm\rangle$$

$$= |C_\pm|^{-2} \langle jm | J_\mp J_\pm | jm\rangle$$

$$\underset{(\because ⑤⑥)}{=} |C_\pm|^{-2} \langle jm | (J^2 - J_z(J_z \pm \hbar) | jm\rangle)$$

$$= |C_\pm|^{-2} \langle jm | \{\hbar^2 j(j+1) - \hbar m(\hbar m \pm \hbar)\} | jm\rangle$$

$$= |C_\pm|^{-2} \hbar^2 \{ j(j+1) - m(m\pm1)\} \underbrace{\langle jm|jm\rangle}_{=1}$$

$$\therefore |C_\pm|^2 = \hbar^2 \{ j(j+1) - m(m\pm1)\}$$

for $+$, $j(j+1) - m(m+1) = j^2 + j - m^2 - m = (j+m+1)(j-m)$

for $-$, $j(j+1) - m(m-1) = j^2 + j - m^2 + m = (j-m-1)(j+m)$ //

$$\Rightarrow \quad |j\, m\pm1\rangle = \frac{1}{\hbar\sqrt{(j\pm m+1)(j\mp m)}} J_\pm |jm\rangle \qquad ⑦$$

(Phase is chosen as "1".)

$$J^2 - J_z^2 = J_x^2 + J_z^2 = \frac{1}{2}(J_+J_- + J_-J_+)$$

$$\left(J^2 = \vec{J}\cdot\vec{J} \underset{④}{=} \frac{1}{2}(J_+J_- + J_-J_+) + J_z^2 \right)$$

$$\langle jm | \underbrace{J^2 - J_z^2}_{= \hbar^2 j(j+1) - \hbar^2 m^2 = \hbar^2\{j(j+1)-m^2\}} | jm \rangle = \frac{1}{2} \langle jm | (J_+J_- + J_-J_+) | jm \rangle$$

$$= \frac{1}{2}\left(\underbrace{\langle jm | J_+}_{(J_-|jm\rangle)^\dagger} J_- | jm \rangle + \underbrace{\langle jm | J_-}_{(J_+|jm\rangle)^\dagger} J_+ | jm \rangle \right)$$

$$= \frac{1}{2}\left(\| J_- | jm \rangle \|^2 + \| J_+ | jm \rangle \|^2 \right) \quad — (*)$$

$$\left(\| |a\rangle \|^2 = \langle a | a \rangle \right)$$

$(*)$: $\| *_1 \|^2 + \| *_2 \|^2 \geq 0$.

$\longrightarrow$ $j(j+1) - m^2 \geq 0 \qquad j(j+1) \geq m^2$

$-\sqrt{j(j+1)} \leq m \leq \sqrt{j(j+1)}$ restrincted （制限されている）.

これまでの話から,

$$J_\pm |jm\rangle \longrightarrow |j\,m\pm1\rangle$$

$$J_\pm^2 |jm\rangle \propto |j\,m\pm2\rangle$$

- $|j\ m\pm k\rangle \propto J_\pm^k |jm\rangle$ can be zero.

$$\longleftrightarrow \begin{cases} J_+ |j\ m_{max}\rangle = 0 \\ J_- |j\ m_{min}\rangle = 0 \end{cases}$$

⑦ : $|j\ m_{max}+1\rangle\, \hbar\sqrt{(j+m_{max}+1)(j-m_{max})}$

$= J_+ |j\ m_{max}\rangle \qquad \left(\begin{matrix} m_{max} \geq 0 \\ j - m_{max} = 0 \end{matrix} \right.$

- $m_{max} = j \qquad J_+ |jj\rangle = 0$

$m_{min} = -|m_{min}|$

$$|j\ m_{min}-1\rangle\, \hbar\sqrt{(j-m_{min}+1)(\underbrace{j+m_{min}}_{=0})} = J_- |j\ m_{min}\rangle$$

$$\left(-\sqrt{j(j+1)} \leq m \leq \sqrt{j(j+1)} \qquad j < \cdots < j+1 \right)$$

$j + m_{min} = 0 \qquad m_{min} = -j$

- $m_{min} = -j \quad , \quad J_- |j, -j\rangle = 0$

$$
\left.
\begin{aligned}
&|j\,j\rangle \\
&|j\,j-1\rangle \propto J_-|j\,j\rangle \\
&|j\,j-2\rangle \propto J_-^2|j\,j\rangle \\
&\qquad\vdots \\
&|j\,-j\rangle \propto J_-^{2j}|j\,j\rangle
\end{aligned}
\right\}
$$

有限回が終了すると仮定

(無限回となると授業で扱う枠を超える)

$$
\rightarrow \quad
\begin{array}{ll}
j & \\
j-1 & \\
j-2 & \\
\vdots & \\
j-k & k = 0, 1, 2 \cdots \\
\vdots & \\
-j & \text{となる } k \text{ の存在}
\end{array}
$$

ex) $j = 0.3$

$$
\begin{aligned}
j &= 0.3 \\
j-1 &= -0.7 \\
j-2 &= -1.7 \\
&\vdots
\end{aligned}
$$

$-j = -0.3$ が存在しない.

$\rightarrow$ $j = 0.3$ では不適.

従って, $\exists k = 0, 1, 2, \cdots$

$2j = k \quad \rightarrow \quad j = \frac{k}{2} \quad \left(0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3 \cdots \right)$

j : integer ≥ 0 or half odd integer $\frac{1}{2}, \frac{3}{2}, \frac{5}{2} \cdots$

$m = \underbrace{j, j-1, j-2, \cdots, -j}_{2j+1}$: Quatization

$j = 0$	$m = 0$
$j = \frac{1}{2}$	$m = \frac{1}{2}, -\frac{1}{2}$
$j = 1$	$m = 1, 0, -1$
$j = \frac{3}{2}$	$m = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$

$\vdots$