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前回の続き (${E_3}$に連続変形できる ～ のくだり)

(ex) xy 回転

$$R=\begin{pmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix} \quad \det R = 1$$

$$\theta \to 0 \text{ で } R = E_3$$

★ infinitismal rotation (無限小回転)

$R \sim E_3$, $\det R = 1$ と近いというなら

$$\boxed{R \sim E + \delta R}$$

回転なので $R\tilde{R} = E_3$

$$(E_3 + \delta R)(\tilde{E}_3 + \delta\tilde{R}) = E_3$$

$$E_3 + \delta R + \delta\tilde{R} + (\underbrace{\delta\tilde{R}\,\delta R}_{\text{無視}}) = E_3$$

$$\Rightarrow \delta R + \delta\tilde{R} = O = \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$\boxed{\delta\tilde{R} = -\delta R}$$ 反対称 anti-symmetric

$$(\delta\tilde{R})_{ij} = \delta R_{ji} = -\delta R_{ij}$$

$$i = j \Rightarrow \delta R_{ii} = 0$$

$$\therefore \delta R = \begin{pmatrix}0 & A & B\\ -A & 0 & C\\ -B & -C & 0\end{pmatrix} \text{ の形}$$

$$\varepsilon_{ijk} = \begin{cases} 1 & (ijk) = (1\,2\,3)(2\,3\,1)(3\,1\,2)\\ -1 & (ijk) = (2\,1\,3)(1\,3\,2)(3\,2\,1)\\ 0 & \text{other wise}\end{cases}$$

$$\boxed{\varepsilon_{ijk} = -\varepsilon_{jik} = -\varepsilon_{kji} = -\varepsilon_{ikj}}$$

完全反対称 completely anti-symmetric

ex) $\varepsilon_{ijj} = -\varepsilon_{ijj} \Rightarrow 2\varepsilon_{ijj} = 0$

$$\boxed{\varepsilon_{ijk} = 0 \quad \text{if} \quad i=j \text{ or } j=k \text{ or } k=i}$$

$$(\delta R)_{ij} \overset{\text{定ギ}}{\equiv} -\underbrace{\varepsilon_{ijk}\,\delta w_k}_{\text{sum}} \qquad \delta w:\text{real}$$

$$\begin{pmatrix}\delta w_1\\ \delta w_2\\ \delta w_3\end{pmatrix} = \delta\vec{w}$$

& まとめ

if Hamiltonian is invariant for — conserved

- the times → Energy
- space tlans → Momentum
- rotation → Angular momentum

$\vec{r}$ を無限小回転させた $\vec{r}'$

$$(\vec{r}')_i = (R\vec{r})_i = \underbrace{R_{ij} r_j}_{\text{和}} = (E + \delta R)_{ij} r_j$$

$$= r_j + \delta R_{ij} r_j$$

$$= r_j - \varepsilon_{ijk}\,\delta w_k\, r_j$$

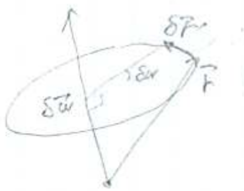
$$(\delta r)_i = r_i' - r_i$$

$$= -\varepsilon_{ijk}\,\delta w_k\, r_j$$

$$= \varepsilon_{ikj}\, r_j\, \delta w_k$$

$$= (\delta\vec{w} \times \vec{r})_i \qquad (\vec{A}\times\vec{B})_i = \varepsilon_{ijk} A_j B_k$$

$$\boxed{\delta\vec{r} = \delta\vec{w} \times \vec{r}}$$



$$|\delta\vec{r}| = |\delta w|\,|r| \sin\theta$$

波動関数 ψ を操作

$$\psi'(\vec{r}') = \psi(\vec{r})$$
$$= \psi'(R\vec{r})$$
$$\therefore \psi'(\vec{r}) = \psi(R^{-1}\vec{r})$$
$$= \psi(\vec{r} - \underbrace{\delta R\,\vec{r}}_{\delta\vec{r}\text{とおく}})$$

☺ $R \sim E_3 + \delta R$ ⇓ $R\tilde{R} = E$; $R^{-1} \sim E_3 - \delta R$ ($\tilde{R}^{-1}$)

$$\delta\psi = \psi'(\vec{r}) - \psi(\vec{r})$$
$$= \psi(\vec{r} - \delta\vec{r}) - \psi(\vec{r})$$
$$\sim \psi(\vec{r}) - \underbrace{\delta\vec{r}\cdot\vec{\nabla}\psi}_{\text{3dim テーラー展開}} - \psi(\vec{r})$$
$$= -\delta\vec{r}\cdot\vec{\nabla}\psi$$
$$\vec{p} = \frac{\hbar}{i}\vec{\nabla}$$
$$= -\frac{i}{\hbar}\,\delta\vec{r}\cdot\vec{p}\,\psi$$
$$= -\frac{i}{\hbar}(\delta\vec{w}\times\vec{r})\cdot\vec{p}\,\psi$$
$$= -\frac{i}{\hbar}(\vec{r}\times\vec{p})\cdot\delta\vec{w}\,\psi$$

☺ $(\vec{A}\times\vec{B})\cdot\vec{C}$ は順≥で同じ

$\vec{L} = \vec{r}\times\vec{p}$ Angular momentum

$$\therefore \boxed{\delta\psi = -\frac{i}{\hbar}\,\delta\vec{w}\cdot\vec{L}}$$

cf) $\psi' = e^{i\delta\lambda G/\hbar}\psi$ Unitary operator

$$\boxed{\delta\psi' = i\delta\lambda G/\hbar}$$

$\delta\lambda : \delta\vec{w}$

$G : \vec{L}$ generator of the rotation (回転の母関数)

transformation low (変換則)

物理量 $O \rightarrow O' = UOU^{\dagger}$

$$\langle\psi|O|\psi\rangle = \langle\psi'|O'|\psi'\rangle \qquad \psi' = U\psi$$

$$= \langle\psi|\underbrace{U^{\dagger}O'U}_{O}|\psi\rangle$$

$$\delta O = O' - O$$
$$= (1 + i\delta\lambda G/\hbar)O(1 - i\delta\lambda G/\hbar) - O$$
$$= i\delta\lambda/\hbar\,[G, O]$$

(1) $O \Rightarrow \vec{r}$ を観測する。無限小回転で

$$\begin{cases} U = e^{-\delta\vec{\omega}\cdot\vec{L}/\hbar} \\ G = -\vec{L} \end{cases}$$

$$\delta\vec{r} = [-i\delta\vec{\omega}\cdot\vec{L}/\hbar, \vec{r}]$$
$$= -\frac{i}{\hbar}[\delta\vec{\omega}\cdot\vec{L}, \vec{r}]$$
$$\delta r_i = -\frac{i}{\hbar}[\delta\omega_j \overset{\text{和}}{L_j}, r_i]$$
$$= -\frac{i}{\hbar}[\delta\omega_j \varepsilon_{jab} r_a p_b, r_i]$$
$$= -\frac{i}{\hbar}(\delta\omega_j \varepsilon_{jab} r_a)[p_b, r_i]$$
$$[r_a, p_b] = \delta_{ab}\, i\hbar$$
$$= -(\delta\omega_j \varepsilon_{jab} r_a)\delta_{ib}$$
$$= -\delta\omega_j \varepsilon_{jai} r_a$$
$$= -\varepsilon_{jai}\delta\omega_j r_a$$
$$= -\varepsilon_{ija}\delta\omega_j r_a$$
$$= -(\delta\vec{\omega}\times\vec{r})_i$$
$$\therefore \boxed{\delta\vec{r} = -\delta\vec{\omega}\times\vec{r}}$$

(2) $\vec{p}$ を観測る。

$$\delta\vec{p} = [-i\delta\vec{\omega}\cdot\vec{L}/\hbar, \vec{p}]$$
$$\delta p_i = -\frac{i}{\hbar}[\delta\omega_j \varepsilon_{jab} r_a p_b, p_i]$$
$$\left([AB, C] = A[B, C] + [A, C]B\right)$$
$$= -\frac{i}{\hbar}\delta\omega_j \varepsilon_{jab}[r_a, p_i]p_b$$
$$= -\frac{i}{\hbar}\delta\omega_j \varepsilon_{jab}\overset{\text{クロネッカーの}}{\delta_{ai}}\, i\hbar\, p_b$$
$$= \delta\omega_j \varepsilon_{jib} p_b$$
$$= \varepsilon_{jib}\delta\omega_j p_b$$
$$= -\varepsilon_{ijb}\delta\omega_j p_b$$
$$= -(\delta\vec{\omega}\times\vec{p})_i$$
$$\therefore \boxed{\delta\vec{p} = -\delta\vec{\omega}\times\vec{p}}$$

(3) $\vec{L}$ を観測する。

$$\delta\vec{L} = [-i\delta\vec{\omega}\cdot\vec{L}/\hbar, \vec{L}]$$
$$\delta L_i = -\frac{i}{\hbar}[\delta\omega_j L_j, L_i]$$
$$= -\frac{i}{\hbar}\delta\omega_j[L_j, L_i]$$
$$= \frac{i}{\hbar}\delta\omega_j[L_i, L_j]$$
$$= -\delta\omega_j \varepsilon_{ijk} L_k$$
$$= -\varepsilon_{ijk}\delta\omega_j L_k$$
$$\therefore \boxed{\delta\vec{L} = -\delta\vec{\omega}\times\vec{L}}$$

$$[L_i, L_j] = [\varepsilon_{iab} r_a p_b, \varepsilon_{jcd} r_c p_d]$$
$$= \varepsilon_{iab}\varepsilon_{jcd}[r_a p_b, r_c p_d]$$

$$[r_a p_b, r_c p_d]$$
$$= r_a[p_b, r_c p_d] + [r_a, r_c p_d]p_b$$
$$= -r_a[r_c p_d, p_b] - p_b[r_c p_d, r_a]$$
$$= -r_a[r_c, p_b]p_d - r_c p_b[p_d, r_a]$$
$$= -r_a\underbrace{[r_c, p_b]}_{\delta_{cb}\, i\hbar}p_d + r_c p_b\underbrace{[r_a, p_d]}_{\delta_{ad}\, i\hbar}$$

$$= -i\hbar\varepsilon_{iab}\varepsilon_{jbd} r_a p_d + i\hbar\varepsilon_{iab}\varepsilon_{jca} r_c p_b$$
$$= i\hbar\varepsilon_{iab}\varepsilon_{jdb} r_a p_d - i\hbar\varepsilon_{iba}\varepsilon_{jca} r_c p_b$$
$$= i\hbar(\delta_{ij}\delta_{ad} - \delta_{id}\delta_{aj}) r_a p_d - i\hbar(\delta_{ij}\delta_{bc} - \delta_{ic}\delta_{bj}) r_c p_b$$
$$= i\hbar(\delta_{ij}\vec{r}\cdot\vec{p} - r_j p_i) - i\hbar(\delta_{ij}\vec{r}\cdot\vec{p} - r_i p_j)$$
$$= i\hbar(r_i p_j - r_j p_i) = i\hbar\varepsilon_{ijk}L_k$$

これらの観察から、transformation low

$$\delta\vec{V} = -\delta\vec{\omega}\times\vec{V} \qquad \vec{V} = \vec{r}, \vec{p}, \cdots$$

$$\vec{V} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \quad \text{vector operator}$$

$K = \frac{\vec{p}^2}{2m}$: kinetic energy (運動E_g)

$\delta K = \frac{1}{2m}(\vec{p}\cdot\delta\vec{p} + \delta\vec{p}\cdot\vec{p})$

$= -\frac{\vec{p}}{m}(\delta\vec{\omega}\times\vec{p})$

$= -\frac{1}{m}\delta\vec{\omega}(\vec{p}\times\vec{p}) = 0$.

$\delta K = [-i\delta\vec{\omega}\cdot\vec{L}/\hbar, \frac{\vec{p}^2}{2m}]$

$= [-i\delta\vec{\omega}\cdot\vec{L}/\hbar, \frac{1}{2m}\vec{p}\cdot\vec{p}]$

$\delta\vec{p}$

$= \delta\vec{p}\cdot\vec{p} + \vec{p}\cdot\delta\vec{p}$.

$\delta K = 0$ scalar operator
vector 演算子の内積はスカラー

ex) $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2\vec{r}^2$ (3D hermoric oscilation)

内積

$\delta H = 0$ invaliant of the rotation