量子力学3 6/9　　201310873 田村 啓

授業の記録【事実・知識】

☆ Addition of angular momentum
合成

spin $(S=\frac{1}{2})$ $\vec{S}=\vec{S}_1+\vec{S}_2$

$\vec{S}_1 : S=\frac{1}{2}$ $\vec{S}_1^2 = S(S+1) = \frac{1}{2}(\frac{1}{2}+1)=\frac{3}{4}$

$\vec{S}^2|S,M\rangle = \hbar^2 S(S+1)|S,M\rangle$ $(j\,m)=(S,M)$ (歴史的名残)

$\| S_x^2+S_y^2+S_z^2 = \frac{1}{2}(S_+S_-+S_-S_+)+S_z^2$

$S_z|S,M\rangle = \hbar M|S\,M\rangle$, $S=\frac{1}{2}$

$M=-S,-S+1,\dots S$

$|S,M\rangle = |M\rangle$ $-\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$

$S_z|M\rangle = \hbar M|M\rangle$ $-\frac{1}{2},\frac{1}{2}=\downarrow,\uparrow$

S_1, S_2

$S_1^z|\uparrow\rangle_1 = \frac{\hbar}{2}|\uparrow\rangle_1$ $S_2|\uparrow\rangle_2 = \frac{\hbar}{2}|\uparrow\rangle_2$

$S_1^z|\downarrow\rangle_1 = -\frac{\hbar}{2}|\downarrow\rangle_1$ $S_2|\downarrow\rangle_2 = -\frac{\hbar}{2}|\downarrow\rangle_2$

$\vec{S}=\vec{S}_1+\vec{S}_2 \to [S_\alpha^i, S_\beta^j] = i\hbar\varepsilon^{ijk}S_\alpha^k\delta_{\alpha\beta}$

Angular momentum $\Rightarrow [S_i, S_j] = i\hbar\varepsilon_{ijk}S_k$

$\vec{S}^2 = \vec{S}_1^2+\vec{S}_2^2+2\vec{S}_1\cdot\vec{S}_2 = \frac{3}{4}+\frac{3}{4}+2\vec{S}_1\cdot\vec{S}_2$
$= \frac{3}{2}+2\vec{S}_1\cdot\vec{S}_2$

$S_\pm = S_1^\pm + S_2^\pm$ $S_z = S_1^z+S_2^z$

$\vec{S}^2|SM\rangle = \hbar^2 S(S+1)|SM\rangle$
$S_z|SM\rangle = \hbar M|SM\rangle$

$|S,M\rangle$? $S=?$ $M=-S,\dots,S$

$|\uparrow\uparrow\rangle = |\uparrow\rangle_1|\uparrow\rangle_2$

$S^+|\uparrow\uparrow\rangle = (S_1^+ + S_2^+)|\uparrow\rangle_1|\uparrow\rangle_2$
$= \underbrace{S_1^+|\uparrow\rangle_1}_{0}|\uparrow\rangle_2 + |\uparrow\rangle_1\underbrace{S_2^+|\uparrow\rangle_2}_{0} = 0$

$|\uparrow\uparrow\rangle = |S, M=S\rangle$

$S_z|\uparrow\uparrow\rangle = (S_1^z+S_2^z)|\uparrow\rangle_1|\uparrow\rangle_2$
$= \underbrace{S_1^z|\uparrow\rangle_1}_{\frac{\hbar}{2}|\uparrow\rangle_1}|\uparrow\rangle_2 + |\uparrow\rangle_1\underbrace{S_2^z|\uparrow\rangle_2}_{\frac{\hbar}{2}|\uparrow\rangle_2}$
$= \frac{\hbar}{2}(|\uparrow\rangle_1|\uparrow\rangle_2 + |\uparrow\rangle_1|\uparrow\rangle_2)$
$= \hbar|\uparrow\rangle_1|\uparrow\rangle_2$
$= \hbar|\uparrow\uparrow\rangle$

メモ【考え・気持ち・自主学習】

$\overset{S}{\|}$
$|S\,M\rangle = |\uparrow\uparrow\rangle$

$S_z|\uparrow\uparrow\rangle = \hbar|\uparrow\uparrow\rangle$
$= \hbar M|\uparrow\uparrow\rangle$ $M=1,\ S=1$

$|S=1, M=1\rangle = |\uparrow\uparrow\rangle$

ここで公式
$J_-|j\,m\rangle = \hbar\sqrt{(j+m)(j-m+1)}\,|j\ m-1\rangle$
$J_+|j\,m\rangle = \hbar\sqrt{(j+m+1)(j-m)}\,|j\ m+1\rangle$
を書いておく

$S_-|1,1\rangle = \hbar\sqrt{2\cdot 1}\,|1\,0\rangle$
$\therefore |1\,0\rangle = \frac{1}{\hbar\sqrt{2}}S_-|\uparrow\uparrow\rangle$

$S=\frac{1}{2}$, $M=\frac{1}{2}$ $S_1^-|\uparrow\rangle_1 = \hbar\sqrt{(\frac{1}{2}+\frac{1}{2})(\frac{1}{2}-\frac{1}{2}+1)}|\downarrow\rangle_1 = \hbar|\downarrow\rangle_1$

$S_2^-|\uparrow\rangle_2 = \hbar|\downarrow\rangle_2$

$|1\,0\rangle = \frac{1}{\hbar\sqrt{2}}S_-|\uparrow\uparrow\rangle$
$= \frac{1}{\hbar\sqrt{2}}(S_1^-+S_2^-)|\uparrow\rangle_1|\uparrow\rangle_2$
$= \frac{1}{\hbar\sqrt{2}}\{S_1^-|\uparrow\rangle_1|\uparrow\rangle_2 + |\uparrow\rangle_1\underbrace{S_2^-|\uparrow\rangle_2}_{\hbar|\downarrow\rangle_2}\}$
$= \frac{1}{\sqrt{2}}(|\downarrow\rangle_1|\uparrow\rangle_2 + |\uparrow\rangle_1|\downarrow\rangle_2)$
$= \frac{1}{\sqrt{2}}(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle)$

$S_-|1\,0\rangle = \hbar\sqrt{(1+0)(1-0+1)}\,|1\,-1\rangle = \hbar\sqrt{2}|1\,-1\rangle$

$|1\,-1\rangle = \frac{1}{\hbar\sqrt{2}}S_-|1\,0\rangle = \frac{1}{\hbar\sqrt{2}}S_-\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle)$
$= \frac{1}{2\hbar}(S_1^-+S_2^-)(\underbrace{|\uparrow\downarrow\rangle}_{|\uparrow\rangle_1|\downarrow\rangle_2}+\underbrace{|\downarrow\uparrow\rangle}_{|\downarrow\rangle_1|\uparrow\rangle_2})$
$= \frac{1}{2\hbar}(\underbrace{S_1^-|\uparrow\rangle_1}_{\hbar|\downarrow\rangle_1}|\downarrow\rangle_2 + \underbrace{S_1^-|\downarrow\rangle_1}_{0}|\uparrow\rangle_2 + |\uparrow\rangle_1\underbrace{S_2^-|\downarrow\rangle_2}_{0} + |\downarrow\rangle_1\underbrace{S_2^-|\uparrow\rangle_2}_{\hbar|\downarrow\rangle_2})$
$= \frac{1}{2\hbar}(\hbar|\downarrow\rangle_1|\downarrow\rangle_2 + \hbar|\downarrow\rangle_1|\downarrow\rangle_2)$
$= |\downarrow\rangle_1|\downarrow\rangle_2$
$= |\downarrow\downarrow\rangle$

まとめ
$|1\,1\rangle = |\uparrow\uparrow\rangle$
$|1\,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$
$|1\,-1\rangle = |\downarrow\downarrow\rangle$
S $M=-S$ $S_-|\downarrow\downarrow\rangle = 0$

$|S\,M\rangle$ $S=1,\ M=1,0,-1$ の3つを見つけた

$|\uparrow\uparrow\rangle = |\uparrow\rangle_1|\uparrow\rangle_2$ $|\uparrow\downarrow\rangle = |\uparrow\rangle_1|\downarrow\rangle_2$
$|\downarrow\downarrow\rangle = |\downarrow\rangle_1|\downarrow\rangle_2$ $|\downarrow\uparrow\rangle = |\downarrow\rangle_1|\uparrow\rangle_2$

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○ $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$ orthonormalized
規格直交系

$\alpha, \beta = \uparrow, \downarrow$

$$\delta_{\alpha\alpha'}\delta_{\beta\beta'} = \langle\alpha\beta|\alpha'\beta'\rangle$$
$$= {}_1\langle\alpha|{}_2\langle\beta|(|\alpha'\rangle_1|\beta'\rangle_2)$$
$$= {}_1\langle\alpha|\alpha'\rangle_1\, {}_2\langle\beta|\beta\rangle_2$$

$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) = |X\rangle$ とする

$$\langle 11|X\rangle = \frac{1}{\sqrt{2}}\langle\uparrow\uparrow|(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$
$$= \frac{1}{\sqrt{2}}(\langle\uparrow\uparrow|\uparrow\downarrow\rangle - \langle\uparrow\uparrow|\downarrow\uparrow\rangle) = 0$$
$$= \frac{1}{\sqrt{2}}(\underbrace{\langle\uparrow|\uparrow\rangle_1\langle\uparrow|\downarrow\rangle_2}_{0} + \underbrace{\langle\uparrow|\downarrow\rangle_1\langle\uparrow|\uparrow\rangle_2}_{0})$$

$\langle 1\,{-1}|X\rangle = 0$

$$\langle 10|X\rangle = \frac{1}{2}(\langle\uparrow\downarrow| + \langle\downarrow\uparrow|)(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$
$$= \frac{1}{2}(\langle\uparrow\downarrow|\uparrow\downarrow\rangle - \langle\downarrow\uparrow|\downarrow\uparrow\rangle - \underbrace{\langle\uparrow\downarrow|\downarrow\uparrow\rangle}_{0} + \underbrace{\langle\downarrow\uparrow|\uparrow\downarrow\rangle}_{0})$$
$$= 0$$

$$(|\uparrow\uparrow\rangle, \tfrac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), \tfrac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle), |\downarrow\downarrow\rangle)$$
$$= (|11\rangle, |10\rangle, |X\rangle, |1\,{-1}\rangle)$$
$$= (|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

base change 基底変換

では $|X\rangle$ とは…?

$$S_z|X\rangle = (S_1^z + S_2^z)\frac{1}{\sqrt{2}}(|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2)$$
$$= \frac{1}{\sqrt{2}}(\underbrace{S_1^z|\uparrow\rangle_1|\downarrow\rangle_2}_{=\frac{\hbar}{2}} + \underbrace{|\uparrow\rangle_1 S_2^z|\downarrow\rangle_2}_{-\frac{\hbar}{2}}$$
$$- \underbrace{S_1^z|\downarrow\rangle_1|\uparrow\rangle_2}_{=-\frac{\hbar}{2}} - \underbrace{|\downarrow\rangle_1 S_2^z|\uparrow\rangle_2}_{=\frac{\hbar}{2}})$$
$$= 0$$

$$S_z|X\rangle = \frac{1}{\sqrt{2}}(S_z|\uparrow\downarrow\rangle - S_z|\downarrow\uparrow\rangle)$$
$$= 0 \qquad 0 \qquad 0$$

$\rightarrow |X\rangle$ $\underline{\underline{M=0}}$

$$S_+|X\rangle = (S_1^+ + S_2^+)\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$
$$= \frac{1}{\sqrt{2}}(\underbrace{S_1^+|\uparrow\downarrow\rangle}_{0} - S_1^+|\downarrow\uparrow\rangle = -\hbar|\uparrow\uparrow\rangle$$
$$+ \underbrace{S_2^+|\uparrow\downarrow\rangle}_{+|\uparrow\uparrow\rangle} - \underbrace{S_2^+|\downarrow\uparrow\rangle}_{0})$$
$$= 0$$

M: max $\quad \underline{\underline{S = M = 0}}$

	S	M
$\|11\rangle = \|\uparrow\uparrow\rangle$	1	1
$\|10\rangle = \frac{1}{\sqrt{2}}(\|\uparrow\downarrow\rangle + \|\downarrow\uparrow\rangle)$	1	0
$\|1\,{-1}\rangle = \|\downarrow\downarrow\rangle$	1	−1
$\|00\rangle = \frac{1}{\sqrt{2}}(\|\uparrow\downarrow\rangle - \|\downarrow\uparrow\rangle)$ ($= \|X\rangle$)	0	0

$S = \frac{1}{2}$ spins (2)

$\Longrightarrow S = 1, 0$

$$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$$

↑ state

$$|M_1\rangle_1 |M_2\rangle_2 = |M_1\rangle_1 \otimes |M_2\rangle_2 = \frac{1}{2}(1-1) = 0$$

tensor product, direct product
テンソル積 直積

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ …Clebsch-Gordan coefficient

$S_1 = \frac{1}{2}$, $M_1 = \pm\frac{1}{2}$, $S_2 = \frac{1}{2}$, $M_2 = \pm\frac{1}{2}$

$$|S_1M_1, S_2M_2\rangle = \underbrace{|S_1M_1\rangle}_{2S_1+1} \otimes \underbrace{|S_2M_2\rangle}_{2S_2+1}$$

↓ base change $\quad (2S_1+1)(2S_2+1) = 2\times 2 = 4$

$|SM\rangle$ total angular momentum
全角運動量

$S = 1, 0$

$2S+1 = \begin{cases} 3 \\ 1 \end{cases} \quad 3+1=4$

$\langle SM|S_1M_1\,S_2M_2\rangle$ CG 係数

$$\frac{1}{2}\otimes\frac{1}{2} = 1 \oplus 0$$

$2S+1=3$ triplet 三重項

$2S+1 = 2\cdot 0+1 = 1$ singlet 一重項