理工学群 物理学類 3年 201310875 出口 裕径

★ Algebra of angular momentum
－角運動量の代数－

Rotation : unitory op. $U = e^{-i\delta\vec{\omega}\cdot\vec{L}/\hbar}$

Angular momentum $\vec{L} = \vec{r}\times\vec{p}$, $\vec{p} = \frac{\hbar}{i}\nabla$ について交換関係などを考える。

$$L_i = \varepsilon_{ijk} r_j p_k$$

c.p. cycle permitation 順に入れかえる

tool
$[AB, C] = A[B,C] + [A,C]B$
$[A, BC] = [A,B]C + B[A,C]$

$$\begin{aligned}
[L_x, L_y] &= [yP_z - zP_y,\ zP_x - xP_z] \\
&= [yP_z,\ zP_x - xP_z] - [zP_y,\ zP_x - xP_z] \\
&= [yP_z, zP_x] - [yP_z, xP_z] \\
&\quad - [zP_y, zP_x] + [zP_y, xP_z] \\
&= y[P_z, z]P_x + x[z, P_z]P_y \\
&= -i\hbar yP_x + i\hbar xP_y \\
&= i\hbar(xP_y - yP_x) = i\hbar L_z
\end{aligned}$$

$\therefore \boxed{[L_x, L_y] = i\hbar L_z}$

Lは c.p. より $[L_y, L_z] = i\hbar L_x$
$[L_z, L_x] = i\hbar L_y$

～Another way～

$L_i = \varepsilon_{iab} r_a p_b$, $L_j = \varepsilon_{jcd} r_c p_d$

$$\begin{aligned}
[L_i, L_j] &= \varepsilon_{iab}\varepsilon_{jcd}[r_a p_b, r_c p_d] \\
&= \varepsilon_{iab}\varepsilon_{jcd}\{r_a[p_b, r_c p_d] \\
&\qquad + [r_a, r_c p_d]p_b\} \\
&= \varepsilon_{iab}\varepsilon_{jcd}\{r_a[p_b, r_c]p_d \\
&\qquad + r_c[r_a, p_d]p_b\} \\
&= \varepsilon_{iab}\varepsilon_{jcd}\{r_a(-i\hbar\delta_{bc})p_d \\
&\qquad + r_c(i\hbar\delta_{ad})p_b\} \\
&= -i\hbar\varepsilon_{iab}\varepsilon_{jbd} r_a p_d \\
&\qquad + i\hbar\varepsilon_{iab}\varepsilon_{jca} r_c p_b \\
&= i\hbar\varepsilon_{iab}\varepsilon_{jdb} r_a p_d \\
&\qquad - i\hbar\varepsilon_{iba}\varepsilon_{jca} r_c p_b \\
&= i\hbar(\delta_{ij}\delta_{ad} - \delta_{id}\delta_{aj}) r_a p_d \\
&\qquad - i\hbar(\delta_{ij}\delta_{bc} - \delta_{ic}\delta_{bj}) r_c p_b \\
&= i\hbar(r_i p_j - r_j p_i) \\
&= i\hbar\varepsilon_{ijk} L_k
\end{aligned}$$

同じ

tool
$$\varepsilon_{i\alpha\beta}\varepsilon_{i\gamma\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}$$

$$\begin{aligned}
\varepsilon_{ijk} L_k &= \varepsilon_{ijk}\varepsilon_{kab} r_a p_b \\
&= \varepsilon_{ijk}\varepsilon_{abk} r_a p_b \\
&= (\delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja}) r_a p_b \\
&= r_i p_j - r_j p_i
\end{aligned}$$

Level up !! $\vec{J} = \begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix}$

$$[J_i, J_j] = i\hbar \varepsilon_{ijk} J_k$$

のみみとめたら、何ができるか？？

$\vec{J}^2 = J_x^2 + J_y^2 + J_z^2$ について

$$\boxed{[\vec{J}^2, J_x] = [\vec{J}^2, J_y] = [\vec{J}^2, J_z] = 0}$$

$\because$ $[\vec{J}^2, J_z] = [J_x^2 + J_y^2 + J_z^2, J_z]$

$= [J_x^2 + J_y^2, J_z] + \underset{0}{[J_z^2, J_z]}$

$= J_x[J_x, J_z] + J_y[J_y, J_z] + [J_x, J_z]J_x + [J_y, J_z]J_y$

$= J_x(-i\hbar J_y) + J_y(i\hbar J_x) + (-i\hbar J_y)J_x + (i\hbar)J_x J_y$

$= 0$

c.p..

交換可能 => 同時固有状態がある

$\vec{J}^2 |*\rangle = A|*\rangle$, $J_z|*\rangle = B|*\rangle$

$\vec{J}^2$: Hermite ($\because$ J_x, J_y, J_z Hermite) ?

A, B : real

Lの性質使っているが…

$$U = e^{i\delta w \cdot L/\hbar} \quad \delta w : \text{dimensionles}$$

$[L] = \hbar$

$B \propto \hbar \times$ real number

$\Rightarrow B = \hbar m$ $\quad m$: real

$A \propto \hbar^2 \times$ real number

$\Rightarrow A = \hbar^2 j(j+1)$ $\quad j$: real

とおける。

$\vec{J}^2|j, m\rangle = \hbar^2 j(j+1)|j, m\rangle$

$J_z|j, m\rangle = \hbar m|j, m\rangle$

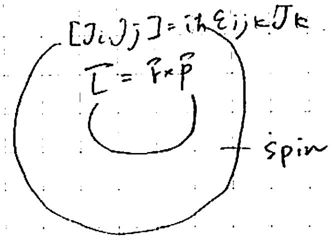
j, m は何だろう??

define $\boxed{J_\pm \equiv J_x \pm iJ_y}$ ladder operators

$J_\pm^\dagger = J_x^\dagger \mp iJ_y^\dagger$

Hermite

$= J_x \mp iJ_y = J_\mp$



ネタバレ

$j = \begin{cases} 0, 1, 2, 3 & \text{non negative integers} \\ \frac{1}{2}, \frac{3}{2}, \frac{5}{2} \cdots & \text{half odd integers} \end{cases}$

非負数

(半奇整数)

$m = -j, \underset{1}{-j+1}, \ldots, j-1, j$

m は $j-(-j-1) = 2j+1$ コ

if) $j=0 \Rightarrow m=0$ (1コ)

$j=1 \Rightarrow m=-1, 0, 1$ (3コ)

$j=2 \Rightarrow m=-2, -1, 0, 1, 2$ (5コ)

⋮

奇数コ

$j=\frac{1}{2} \Rightarrow m=-\frac{1}{2}, \frac{1}{2}$ (2コ)

$j=\frac{3}{2} \Rightarrow m=-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$ (4コ)

偶数コ

☆ $[\vec{J}^2, J_\pm] = [\vec{J}^2, J_x] \pm i[\vec{J}^2, J_y]$

$= 0 \pm 0 = 0$

$[J_+, J_-] = [J_x + iJ_y, J_x - iJ_y]$

$= -i[J_x, J_y] + i[J_y, J_x]$

$= -i \cdot i\hbar J_z + i(-i\hbar)J_z$

$= 2\hbar J_z$

$[J_z, J_\pm] = [J_z, J_x \pm iJ_y]$

$= [J_z, J_x] \pm i[J_z, J_y]$

$= i\hbar J_y \mp i i\hbar J_x$

$= \pm\hbar(J_x \pm iJ_y) = \pm\hbar J_\pm$

$\vec{J}^A, \vec{J}^B$ について.

$$\boxed{\vec{J}^A \cdot \vec{J}^B = \frac{1}{2}(J_+^A J_-^B + J_-^A J_+^B) + J_z^A J_z^B}$$

Important formula

$J_x = \frac{1}{2}(J_+ + J_-)$, $J_y = \frac{1}{2i}(J_+ - J_-)$

$\because$ $\vec{J}^A \cdot \vec{J}^B = J_x^A J_x^B + J_y^A J_y^B + J_z^A J_z^B$

$= \frac{1}{4}(J_+^A + J_-^A)(J_+^B + J_-^B)$

$- \frac{1}{4}(J_+^A - J_-^A)(J_+^B - J_-^B) + J_z^A J_z^B$

$= \frac{1}{2}(J_+^A J_-^B + J_-^A J_+^B) + J_z^A J_z^B$

$\vec{J}^2 = \vec{J} \cdot \vec{J} = \frac{1}{2}(J_+ J_- + J_- J_+) + J_z^2 \quad (*1)$

$[J_+, J_-] = 2\hbar J_z \iff \hbar J_z = \frac{1}{2}(J_+ J_- - J_- J_+)$

$(*2)$

$*1 + *2$

$\vec{J}^2 + \hbar J_z = J_+ J_- + J_z^2$

$\iff \boxed{J_+ J_- = \vec{J}^2 - J_z(J_z - \hbar)}$

$*1 - *2$

$\vec{J}^2 - \hbar J_z = J_- J_+ + J_z^2$

$\iff \boxed{J_- J_+ = \vec{J}^2 - J_z(J_z + \hbar)}$.