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DATE

$$D^{j}_{m'm}(R(\alpha\beta\gamma)) = e^{-i\alpha m'} d^{j}_{m'm}(\beta) e^{-i\gamma m}$$

$$d^{j}_{m'm} = \sum_{k}(-)^{k-m+m'} \frac{\sqrt{(j+m)!(j-m)!(j+m')!(j-m')!}}{(j+m-k)!k!(k-m+m')!(j-k-m')!} \cos^{2j-2k+m-m'}\frac{\beta}{2} \sin^{2k-m+m'}\frac{\beta}{2}$$

$j = l = 0, 1, 2, 3, \ldots$

$$Y_{lm}(\beta\alpha) = C\left\{D^{l}_{m0}(\alpha,\beta)\right\}^{*} \qquad C?$$

$m = l$

$$Y_{ll}(\underset{\theta,\phi}{\beta,\alpha}) = (-)^{l}\frac{1}{\sqrt{2\pi}}\sqrt{\frac{(2l+1)!}{2}}\frac{1}{2^{l}l!}e^{il\alpha}\sin^{l}\beta$$

$$D^{l}_{l0}(\alpha,\beta) = e^{-il\alpha}\sum_{k}(-)^{k+l}\frac{\sqrt{l!l!(2l)!0!}}{(l-k)!k!(k+l)!(-k)!}\cos^{2l-2k-l}\frac{\beta}{2}\sin^{2k+l}\frac{\beta}{2}$$

$k = 0$ のときのみ!

$$= (-)^{l}e^{-i\alpha l}\frac{\sqrt{l!l!(2l)!}}{l!l!}\underbrace{\cos^{l}\frac{\beta}{2}\sin^{l}\frac{\beta}{2}}_{= \frac{1}{2^{l}}\sin^{l}\beta}$$

$$Y_{ll} = D^{l*}_{l0}\frac{1}{\sqrt{2\pi}}\sqrt{\frac{2l+1}{2}} = \sqrt{\frac{2l+1}{4\pi}}D^{l*}_{l0}$$

$$Y_{lm}(\beta,\alpha) = D^{l}_{m0}(\alpha,\beta,\gamma)^{*}\sqrt{\frac{2l+1}{4\pi}}$$

$R_0 = R_0(\alpha_0, \beta_0, \gamma_0)$

$R_1 = R_1(\alpha_1, \beta_1, \gamma_1)$ として、 $R_0^{-1}R_1 \equiv R_2$

$$D^{\ell}_{00}(R_0^{-1}R_1) = \left(D^{\ell}(R_0^{-1})\,D^{\ell}(R_1)\right)_{00}$$
$$= \left(D^{\ell}_{m0}(R_0)\right)^{*} D^{\ell}_{m0}(R_1)$$

$$Y_{\ell m} = \sqrt{\frac{2\ell+1}{4\pi}}\,D^{\ell\,*}_{m0}$$

$$\sqrt{\frac{4\pi}{2\ell+1}}\,Y^{*}_{00}(R_2\hat{z}) = \sum_m Y_{\ell m}(R_0\hat{z})\,Y^{*}_{\ell m}(R_1\hat{z})\frac{2\ell+1}{4\pi}$$

$$\sqrt{\frac{2\ell+1}{4\pi}}\,P_\ell(\cos\beta_2) \qquad (\beta_0\alpha_0) \qquad (\beta_1\alpha_1)$$

$$\sum_m Y_{\ell m}(\beta_0\alpha_0)\,Y_{\ell m}(\beta_1\alpha_1) = \frac{4\pi}{2\ell+1}P_\ell(\cos\beta_2)$$

$R_0R_2 = R_1$, $R_1 = R_0R_2$ 　 $\hat{z}_1 = R_1\hat{z} = R_0R_2\hat{z}$, $\hat{z}_0 = R_0\hat{z}$ 　 $\hat{z}_1$ と $\hat{z}_0$ のなす角 β_2 $=$ $\hat{z}$ と $R_2\hat{z}$ のなす角

球面調和関数の加法定理

$$\frac{4\pi}{2\ell+1}\sum_m Y_{\ell m}(\hat{r})\,Y^{*}_{\ell m}(\hat{r}') = P_\ell(\hat{r}\cdot\hat{r}')$$

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell$$

Legendre 多項式

Legendre 多項式の母関数

$$\frac{1}{\sqrt{1-2x\zeta+\zeta^2}} = \sum_{n=0}^{\infty} P_n(x)\zeta^n$$

$|\zeta = 0 \sim 1$

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{\sqrt{(\vec{r}-\vec{r}')^2}} = \frac{1}{\sqrt{\vec{r}^2+\vec{r}'^2-2\vec{r}\cdot\vec{r}'}} = \frac{1}{\sqrt{r_>^2+r_<^2-2r_>r_<\cos\theta}}$$

$$= \frac{1}{r_>}\frac{1}{\sqrt{1+\left(\frac{r_<}{r_>}\right)^2-2\frac{r_<}{r_>}\cos\theta}}$$

$r_> = \max(|\vec{r}|, |\vec{r}'|)$

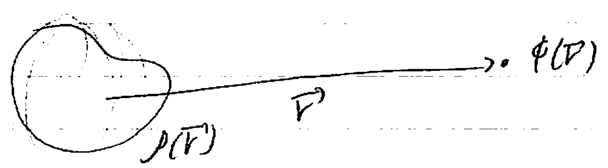
$r_< = \min(|\vec{r}|, |\vec{r}'|)$. $\vec{r}\cdot\vec{r}' = r_>r_<\cos\theta$

$x \leftrightarrow \cos\theta$, $t = \frac{r_<}{r_>}$

$$= \frac{1}{r_>}\sum_{l=0}^{\infty} P_l(\cos\theta)\left(\frac{r_<}{r_>}\right)^l = \sum_{l=0}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta)$$

$$\underset{*}{=} \sum_{lm}\frac{4\pi}{2l+1}Y_{lm}(\hat{r})Y^*_{lm}(\hat{r}')\frac{r_<^l}{r_>^{l+1}}$$

多重極展開



電荷 $\rho(\vec{r}) = 0$. $|\vec{r}| > r_c$

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} \qquad r > r_c,\ |\vec{r}| > |\vec{r}'|$$

$$= \frac{1}{4\pi\varepsilon_0}\int d^3r' \rho(\vec{r}')\sum_{lm}\frac{r'^l}{r^{l+1}}\frac{4\pi}{2l+1}Y_{lm}(\hat{r})Y^*_{lm}(\hat{r}')$$

$$= \sum_{lm}\frac{1}{4\pi\varepsilon_0}\frac{4\pi}{2l+1}\frac{q_{lm}}{r^{l+1}}Y_{lm}(\hat{r}), \quad q_{lm} = \int d^3r'\, r'^l \rho(\vec{r}')Y^*_{lm}(\hat{r}')$$

多重極展開, q_{lm} : 多重極

Clebsch - Gordan係数と回転群

$$|j_1m_1, j_2m_2\rangle = |j_1m_1\rangle \otimes |j_2m_2\rangle = \sum_{jm} |jm\rangle \underbrace{\langle jm | j_1m_1, j_2m_2\rangle}_{\text{CG係数}}$$

$$R|jm\rangle = |jm'\rangle D^j_{m'm}(R)$$

スピンj表現

$$|j_1m_1, j_2m_2\rangle = |jm\rangle\langle jm | j_1m_1 j_2m_2\rangle$$

$$\underbrace{R|j_1m_1, j_2m_2\rangle}_{\substack{= R|j_1m_1\rangle\otimes R|j_2m_2\rangle \\ |j_1m_1'\rangle D^{j_1}_{m_1'm_1}\,|j_2m_2'\rangle D^{j_2}_{m_2'm_2}}} = \underbrace{R|jm\rangle}_{|jm'\rangle D^j_{m'm}}\langle jm | j_1m_1, j_2m_2\rangle$$

$$|j_1m_1', j_2m_2'\rangle D^{j_1}_{m_1'm_1} D^{j_2}_{m_2'm_2} = |jm'\rangle D^j_{m'm}\langle jm | j_1m_1, j_2m_2\rangle$$

$$D^{j_1}_{m_1'm_1} D^{j_2}_{m_2'm_2} = \underbrace{\langle j_1m_1', j_2m_2' | jm'\rangle}_{=\langle jm' | j_1m_1', j_2m_2'\rangle} D^j_{m'm} \underbrace{\langle jm | j_1m_1, j_2m_2\rangle}_{=\langle j_1m_1, j_2m_2 | jm\rangle}$$

$j_1 = l_1 \quad j_2 = l_2, \quad m_1 = m_2 = 0$

$$\underbrace{D^{l_1}_{m_1'0} D^{l_2}_{m_2'0}}_{\frac{4\pi}{\sqrt{(2l_1+1)(2l_2+1)}} Y^*_{l_1m_1'} Y^*_{l_2m_2'}} = \langle l_1m_1', l_2m_2' | lm'\rangle \underbrace{D^l_{m'm}}_{\sqrt{\frac{4\pi}{2l+1}} Y^*_{lm'}} \langle l\underset{0}{m} | l_1 0, l_2 0\rangle$$

$$Y_{l_1m_1}(\hat{r}) Y_{l_2m_2}(\hat{r}) = \sum_{lm} \sqrt{\frac{(2l_1+1)(2l_2+1)}{(2l+1)4\pi}} \langle l\,0 | l_1\,0, l_2\,0\rangle Y_{lm}(\hat{r}) \langle l_1m_1, l_2m_2 | lm\rangle$$

$$\int d\hat{r}\, Y^*_{lm}(\hat{r}) Y_{l_1m_1}(\hat{r}) Y_{l_2m_2}(\hat{r}) = \sqrt{\frac{(2l_1+1)(2l_2+1)}{(2l+1)4\pi}} \langle l\,0 | l_1\,0, l_2\,0\rangle \langle l_1m_1, l_2m_2 | lm\rangle$$