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Schwinger Boson　$[a_\alpha, a_\beta^\dagger] = \delta_{\alpha\beta}$
$(\alpha, \beta = \pm)$　$[a_\alpha, a_\beta] = 0$

$\vec{J} \equiv \frac{1}{2} a^\dagger \vec{\sigma} a$ と定義　$a = \begin{bmatrix} a_+ \\ a_- \end{bmatrix}$

$J_x = \frac{1}{2}(a_+^\dagger \; a_-^\dagger) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_+ \\ a_- \end{pmatrix} = \frac{1}{2}(a_+^\dagger a_- + a_-^\dagger a_+)$

$[J_\alpha, J_\beta] = i\varepsilon_{\alpha\beta\gamma} J_\gamma$

$\vec{J}^2 = \frac{1}{2}\hat{n}(\hat{n}+1)$,　$\hat{j} = \frac{1}{2}\hat{n} = \frac{1}{2}(\hat{n}_+ + \hat{n}_-)$

$\hat{n}_+ \equiv a_+^\dagger a_+$,　$\hat{n}_- \equiv a_-^\dagger a_-$

$|\hat{n}_+, \hat{n}_-\rangle = \frac{1}{\sqrt{n_+!}}(a_+^\dagger)^{n_+} \frac{1}{\sqrt{n_-!}}(a_-^\dagger)^{n_-}|0\rangle = \frac{(a_+^\dagger)^{n_+}(a_-^\dagger)^{n_-}}{\sqrt{n_+! \cdot n_-!}}|0\rangle$ と定義

$$\begin{cases} a_+^\dagger a_+ |n_+ \; n_-\rangle = \hat{n}_+ |n_+ \; n_-\rangle = n_+ |n_+ n_-\rangle \\ a_-^\dagger a_- |n_+ \; n_-\rangle = \hat{n}_- |n_+ \; n_-\rangle = n_- |n_+ \; n_-\rangle \end{cases}$$

$J_z = \frac{1}{2}(a_+^\dagger \; a_-^\dagger) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a_+ \\ a_- \end{pmatrix} = \frac{1}{2}(a_+^\dagger a_+ - a_-^\dagger a_-) = \frac{1}{2}(\hat{n}_+ - \hat{n}_-)$

$\Rightarrow J_z |n_+ \; n_-\rangle = \frac{1}{2}(n_+ - n_-)|n_+ \; n_-\rangle \equiv m |n_+ \; n_-\rangle$

$$\begin{cases} m \equiv \frac{1}{2}(n_+ - n_-) \\ j = \frac{1}{2}n \equiv \frac{1}{2}(n_+ + n_-) \end{cases} \text{とした} \Rightarrow \begin{cases} n_+ = j + m \\ n_- = j - m \end{cases}$$

これを用いると

$|j \; m\rangle = \frac{(a_+^\dagger)^{j+m} (a_-^\dagger)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}|0\rangle$　と書き換えることが出来る.

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$R = e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z}$ (α, β, γ) Euler 角 について考える

$R\psi = \psi D(R)$ $D(R)$: R の表現行列

$\psi = (\psi_1, \cdots, \psi_d) = (|j\ -j>,\ |j\ -j+1>,\ \cdots,\ |j\ j-1>,\ |j\ j>)$
$d = 2j+1$

$R(|j\ m>) = (|j\ m>)D = |j\ m'> D_{m'm}$

左から $<j\ m'|$ を作用させると

$$D_{m'm} = <j\ m'|R|j\ m>$$
$$= \underbrace{<j\ m'|e^{-i\alpha J_z}}_{\downarrow} e^{-i\beta J_y} \underbrace{e^{-i\gamma J_z}|j\ m>}_{\downarrow}$$
$$\downarrow \quad \downarrow$$
$$= e^{-i\alpha m'} e^{-i\gamma m} d_{m'm}(\beta)$$

$d_{m'm}(\beta) \equiv <j\ m'|e^{-i\beta J_y}|j\ m>$ とおく.

$D_{m'm}$ を決定するには $d_{m'm}(\beta)$ が決まればよい

$$e^{-i\beta J_y}|j\ m> = e^{-i\beta J_y} \frac{(a_+^\dagger)^{j+m}(a_-^\dagger)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}|0>$$

$$= \frac{1}{\sqrt{(j+m)!\,(j-m)!}} e^{-i\beta J_y} a_+^\dagger e^{i\beta J_y} e^{-i\beta J_y} a_+^\dagger e^{i\beta J_y} e^{-i\beta J_y} a_+^\dagger \cdots a_+^\dagger e^{i\beta J_y}$$
$$\times e^{-i\beta J_y} a_-^\dagger e^{i\beta J_y} e^{-i\beta J_y} a_-^\dagger \cdots a_-^\dagger e^{i\beta J_y} e^{-i\beta J_y}|0>$$
$$\downarrow \quad (e^{i\beta J_y} \cdot e^{-i\beta J_y} = 1 \text{ を間に挟む})$$
$$= \frac{1}{\sqrt{(j+m)!\,(j-m)!}} \underbrace{[e^{-i\beta J_y} a_+^\dagger e^{i\beta J_y}] \cdots [e^{-i\beta J_y} a_+^\dagger e^{i\beta J_y}]}_{j+m_2}$$
$$\times \underbrace{[e^{-i\beta J_y} a_-^\dagger e^{i\beta J_y}] \cdots [e^{-i\beta J_y} a_-^\dagger e^{i\beta J_y}]}_{j-m_2} e^{-i\beta J_y}|0>$$

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ここで $J_y = \frac{1}{2}(a_+^\dagger, a_-^\dagger)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a_+ \\ a_- \end{pmatrix} = \frac{i}{2}(-a_+^\dagger a_- + a_-^\dagger a_+)$

a_+ は上昇演算子 . a_- は下降演算子。

真空 $|0\rangle$ に作用させると $a_\pm|0\rangle$ となる.

つまり $J_y|0\rangle = 0$

$$e^{-i\rho J_y}|0\rangle = \left(1 + (-i\rho)J_y + \frac{1}{2}(-i\rho)^2 J_y^2 + \cdots\right)|0\rangle = |0\rangle$$

↑ テイラー展開すると

$$f(\rho) \equiv e^{-i\rho J_y} a_\pm e^{i\rho J_y} = a_\pm'$$

$$\begin{cases} f(\rho) = e^{-i\rho J_y} a_\pm e^{i\rho J_y} \\ f(0) = a_\pm \end{cases}$$

$$\begin{aligned} f'(\rho) &= -i J_y e^{-i\rho J_y} a_\pm e^{i\rho J_y} + e^{-i\rho J_y} a_\pm i J_y e^{i\rho J_y} \\ &= -i e^{-i\rho J_y}(J_y a_\pm - a_\pm J_y) e^{i\rho J_y} \\ &= -i e^{-i\rho J_y}[J_y, a_\pm] e^{i\rho J_y} \\ &= -i e^{-i\rho J_y}\left(\pm\frac{i}{2}a_\mp\right) e^{i\rho J_y} \\ &= \frac{1}{2} e^{-i\rho J_y}(\pm a_\mp) e^{i\rho J_y} \end{aligned}$$

$$\begin{aligned} [J_y, a_+] &= \frac{i}{2}[-a_+^\dagger a_- + a_-^\dagger a_+, a_+] \\ &= \frac{i}{2}\{[-a_+^\dagger a_-, a_+] + [a_-^\dagger a_+, a_+]\} \\ &= \frac{i}{2}\{-a_+^\dagger[a_-, a_+] - [a_+^\dagger, a_+]a_- + a_-^\dagger[a_+, a_+] + [a_-^\dagger, a_+]a_+\} \\ &= \frac{i}{2}a_- \\ [J_y, a_-] &= \frac{i}{2}[-a_+^\dagger a_- + a_-^\dagger a_+, a_-] \\ &= \frac{i}{2}\{-a_+^\dagger[a_-, a_-] - [a_+^\dagger, a_-]a_- + a_-^\dagger[a_+, a_-] + [a_-^\dagger, a_-]a_+\} \\ &= -\frac{i}{2}a_+ \end{aligned}$$

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$$\frac{d}{d\beta} f(\beta) = \frac{1}{2} e^{-i\beta J_y} (\pm a_\mp) e^{i\beta J_y}$$

$$\begin{cases} f_+ = e^{-i\beta J_y} a_+ e^{i\beta J_y} \\ f_- = e^{-i\beta J_y} a_- e^{i\beta J_y} \end{cases}$$ とおく

$$\frac{d}{d\beta}\begin{bmatrix} f_+ \\ f_- \end{bmatrix} = \frac{1}{2}\begin{bmatrix} f_- \\ -f_+ \end{bmatrix}$$ となる

$$\Rightarrow e^{-i\theta\,\hat{n}\cdot\vec{J}}\, a_\pm\, e^{i\theta\,\hat{n}\cdot\vec{J}} = \left(E_2 \cos\frac{\theta}{2} + i\,\hat{n}\cdot\vec{\sigma}\sin\frac{\theta}{2}\right)\begin{pmatrix} a_+ \\ a_- \end{pmatrix}$$

$$\begin{cases} e^{-i\beta J_y}|j\,m\rangle = \dfrac{1}{\sqrt{(j+m)!\,(j-m)!}} \left[e^{-i\beta J_y} a_+^\dagger e^{i\beta J_y}\right]^{j+m} \left[e^{-i\beta J_y} a_-^\dagger e^{i\beta J_y}\right]^{j-m} e^{-i\beta J_y}|0\rangle \\ e^{-i\beta J_y}|0\rangle = |0\rangle \\ e^{i\beta J_y} a_\pm e^{-i\beta J_y} = \left(E_2 \cos\frac{\beta}{2} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\sin\frac{\beta}{2}\right)\begin{pmatrix} a_+ \\ a_- \end{pmatrix} = \begin{pmatrix} a_+\cos\frac{\beta}{2} + a_-\sin\frac{\beta}{2} \\ -a_+\sin\frac{\beta}{2} + a_-\cos\frac{\beta}{2} \end{pmatrix} \end{cases}$$

$$\Rightarrow e^{-i\beta J_y}|j\,m\rangle = \frac{1}{\sqrt{(j+m)!\,(j-m)!}} \left(a_+^\dagger\cos\tfrac{\beta}{2} + a_-^\dagger\sin\tfrac{\beta}{2}\right)^{j+m} \left(-a_+^\dagger\sin\tfrac{\beta}{2} + a_-^\dagger\cos\tfrac{\beta}{2}\right)^{j-m}|0\rangle$$

$$= \frac{1}{\sqrt{(j+m)!\,(j-m)!}} \sum_k \binom{j+m}{k} (a_+^\dagger\cos\tfrac{\beta}{2})^{j+m-k} (a_-^\dagger\sin\tfrac{\beta}{2})^k \times \sum_\ell \binom{j-m}{\ell} (-a_+^\dagger\sin\tfrac{\beta}{2})^{j-m-\ell} (a_-^\dagger\cos\tfrac{\beta}{2})^\ell |0\rangle$$

$$= \frac{1}{\sqrt{(j+m)!\,(j-m)!}} \sum_{k,\ell} (-1)^{j-m-\ell} \binom{j+m}{k}\binom{j-m}{\ell} \times (\cos\tfrac{\beta}{2})^{j+m-k+\ell} (\sin\tfrac{\beta}{2})^{j-m-\ell+k} (a_+^\dagger)^{2j-k-\ell} (a_-^\dagger)^{k+\ell}|0\rangle$$

$$|j\,m'\rangle = \frac{1}{\sqrt{(j+m')!\,(j-m')!}} (a_+^\dagger)^{j+m'} (a_-^\dagger)^{j-m'}|0\rangle$$

$\langle j\,m'|e^{-i\beta J_y}|j\,m\rangle$ を求めた時に $a_+^\dagger$ と $a_-^\dagger$ の積が同じところのみ残る.

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$$\therefore \begin{cases} j+m' = 2j-k-l \\ j-m' = k+l \end{cases} \Rightarrow j-m'-k-l=0$$

l を消去すると

$$\begin{cases} j-m-l = j-m-(j-m'-k) = -m+m'+k \\ j+m-k+l = j+m-k+(j-m'-k) = 2j-2k+m-m' \\ j-m-l+k = j-m-(j-m'-k)+k = 2k-m+m' \end{cases}$$

$$\therefore d_{m'm} = \langle j\,m' | e^{-i\beta J_y} | j\,m \rangle$$

$$= \frac{1}{\sqrt{(j+m)!\,(j-m)!}} \sum_k (-1)^{-m+m'+k} \left(\cos\frac{\beta}{2}\right)^{2j-2k+m-m'} \left(\sin\frac{\beta}{2}\right)^{2k-m+m'} \binom{j+m}{k} \binom{j-m}{j-m'-k}$$

$$\times \frac{1}{\sqrt{(j+m')!\,(j-m')!}} (j+m')!\,(j-m')!$$

$$\binom{j+m}{k} \to \frac{(j+m)!}{(j+m-k)!\,k!}, \qquad \binom{j-m}{j-m'-k} \to \frac{(j-m)!}{(k+m'-m)!\,(j-m'-k)!}$$

$$= \sum_k \frac{\sqrt{(j+m)!\,(j-m)!\,(j+m')!\,(j-m')!}}{(j+m-k)!\,k!\,(k+m'-m)!\,(j-m'-k)!} (-1)^{-m+m'+k} \left(\cos\frac{\beta}{2}\right)^{2j-2k+m-m'} \left(\sin\frac{\beta}{2}\right)^{2k-m+m'}$$

ここの k は 階乗部分が負の値にならない範囲