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量子力学3

既約テンソル演算子

対称性　　回転操作　　不変 → スカラー
成分あり
ある特定の変換則に従う
↳ ベクトル、テンソル

ベクトル演算子

$[L_i, V_j] = i\hbar \varepsilon_{ijk} V_k$

L_i：無限小回転の生成子

$$\vec{V} = \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

$\vec{V}$：$\vec{L}$ 角運動量
$\vec{P}$ 運動量
$\vec{r}$ 位置ベクトル

テンソルとは？
慣性テンソル
$r_i r_j \cdots$

球面調和関数

$$\vec{L} Y_{lm} f = (\vec{L} Y_{lm}) f + Y_{lm} \vec{L} f$$

$$L_\pm = L_x \pm i L_y$$

$$L_\pm Y_{lm} = L_\pm |l, m\rangle = \hbar\sqrt{(l \mp m)(l \pm m + 1)}\, |l, m\pm 1\rangle = \hbar\sqrt{(l \mp m)(l \pm m + 1)}\, Y_{l, m\pm 1}$$

$$[L_\pm, Y_{lm}] f = L_\pm (Y_{lm} f) - Y_{lm}(L_\pm f) = \hbar\sqrt{(l \mp m)(l \pm m + 1)}\, Y_{l, m\pm 1} f$$

f：任意の関数

$$\begin{cases} [L_\pm, Y_{lm}] = \hbar\sqrt{(k \mp q)(k \pm q + 1)}\, T^k_{q\pm 1} & \cdots ① \\ [L_z, Y_{lm}] = \hbar q\, T^k_q & \cdots ② \end{cases}$$

T^k_q：k階の既約テンソル演算子

$\left(\begin{array}{l} l \to k \\ m \to q \end{array} \quad q : \underbrace{-k \cdots k}_{2k+1 \text{コ}} \right)$

$k = 1$ のとき

$$[J_\pm, T^1_q] = \hbar\sqrt{(2 \pm q)(1 \mp q)}\, T^1_{q\pm 1} \qquad (\because ①)$$

$$\left(\begin{array}{ll} [J_+, T^1_1] = 0 & , [J_-, T^1_1] = \hbar\sqrt{2}\, T^1_0 \\ [J_+, T^1_0] = \hbar\sqrt{2}\, T^1_1 & , [J_-, T^1_0] = \hbar\sqrt{2}\, T^1_{-1} \\ [J_+, T^1_{-1}] = \hbar\sqrt{2}\, T^1_0 & , [J_-, T^1_{-1}] = 0 \\ & \\ [J_z, T^1_1] = \hbar q\, T^1_1 & \\ [J_z, T^1_0] = \hbar q\, T^1_0 & \\ [J_z, T^1_{-1}] = \hbar q\, T^1_{-1} & \end{array} \right)$$

ベクトル演算子 $\vec{J}$

$$[J_+, J_+] = 0 \quad , \quad [J_-, J_+] = -2\hbar J_z$$
$$[J_+, J_z] = -\hbar J_+ \quad , \quad [J_-, J_z] = \hbar J_-$$
$$[J_+, T^1_0] = \hbar\sqrt{2}\, T^1_1$$
$$[J_+, J_-] = 2\hbar J_z \quad , \quad [J_-, J_-] = 0$$

$$\left(\begin{aligned} -\sqrt{2}\, T^1_1 &= J_+ \\ T^1_0 &= J_z \\ \sqrt{2}\, T^1_{-1} &= J_- \end{aligned}\right.$$

$$[J_i, J_j] = i\hbar\, \varepsilon_{ijk} J_k$$
$$J_\pm = J_x \pm i J_y$$

$$-\frac{1}{\sqrt{2}} J_+ = T^1_1 \ , \ \frac{1}{\sqrt{2}} J_- = T^1_{-1} \ , \ J_z = T^1_0$$

T^1_q : 1階の既約テンソル
$q = -1, 0, 1$

ベクトル演算子

$$[J_i, V_j] = i\hbar\, \varepsilon_{ijk} V_k$$

$$T^1_1 = -\frac{1}{\sqrt{2}}(V_x + iV_y)$$
$$T^1_0 = V_z$$
$$T^1_{-1} = \frac{1}{\sqrt{2}}(V_x - iV_y)$$

積 → 高階テンソル

cf. $\boxed{T_{ij} = r_i r_j}$

☆ 既約テンソル演算子の積

$$T^k_q = \sum_{q_1, q_2} T^{k_1}_{q_1} T^{k_2}_{q_2} \underbrace{\langle k_1 q_1, k_2 q_2 | k q \rangle}_{\text{Clebsch-Gordan係数}}$$

$T^{k_1}_{q_1} \uparrow Y^{k_1}_{q_1}$, $T^{k_2}_{q_2} \uparrow Y^{k_2}_{q_2}$

$$\left(|kq\rangle = \sum_{q_1, q_2} \underbrace{|k_1 q_1\rangle |k_2 q_2\rangle}_{|k_1 q_1, k_2 q_2\rangle} \langle k_1 q_1, k_2 q_2 | k q \rangle \right)$$

$$[J_z, T^k_q] = \sum_{q_1, q_2} [J_z, T^{k_1}_{q_1} T^{k_2}_{q_2}] \langle k_1 q_1, k_2 q_2 | k q \rangle$$

$$T^{k_1}_{q_1} \underbrace{[J_z, T^{k_2}_{q_2}]}_{\hbar q_2 T^{k_2}_{q_2}} + \underbrace{[J_z, T^{k_1}_{q_1}]}_{\hbar q_1 T^{k_1}_{q_1}} T^{k_2}_{q_2}$$

$$= \sum_{q_1, q_2} \hbar \underbrace{(q_1 + q_2)}_{= q} T^{k_1}_{q_1} T^{k_2}_{q_2} \langle k_1 q_1, k_2 q_2 | k q \rangle$$

$$\Rightarrow \quad [J_z, T^k_q] = \hbar q T^k_q$$

$$\hbar\sqrt{(k_2\mp q_2)(k_2\pm q_2+1)}\;T^{k_2}_{q_2\pm1}$$

$$[J_\pm, T^k_q] = \sum_{q_1,q_2}\Big\{T^{k_1}_{q_1}[J_\pm, T^{k_2}_{q_2}]\langle k_1q_1,k_2q_2|kq\rangle \longleftarrow q_2\to q_2\mp1$$

$$+[J_\pm, T^{k_1}_{q_1}]T^{k_2}_{q_2}\langle k_1q_1,k_2q_2|kq\rangle\Big\} \longleftarrow q_1\to q_1\mp1$$

$$\hbar\sqrt{(k_1\mp q_1)(k_1\pm q_1+1)}\;T^{k_1}_{q_1\pm1}$$

$$=\sum_{q_1q_2}\Big\{T^{k_1}_{q_1}\hbar\sqrt{\{k_2\mp(q_2\mp1)\}\{k_2\pm(q_2\mp1)+1\}}\;T^{k_2}_{q_2}\langle k_1q_1,k_2q_2\mp1|kq\rangle$$

$$+\hbar\sqrt{\{k_1\mp(q_1\mp1)\}\{k_1\pm(q_1\mp1)+1\}}\;T^{k_1}_{q_1}T^{k_2}_{q_2}\langle k_1q_1\mp1,k_2q_2|kq\rangle\Big\}$$

$$=\sum_{q_1,q_2}T^{k_1}_{q_1}T^{k_2}_{q_2}\Big\{\hbar\sqrt{(k_2\mp q_2+1)(k_2\pm q_2)}\langle k_1q_1,k_2q_2\mp1|kq\rangle$$

$$+\hbar\sqrt{(k_1\mp q_1+1)(k_1\pm q_1)}\langle k_1q_1\mp1,k_2q_2|kq\rangle\Big\}$$

{ } 内の式 → CG係数の関係式

$$J_{1\mp}|k_1q_1\rangle = \bigcirc|k_1q_1\mp1\rangle$$

$$\{\ \} \to \underbrace{\sqrt{(k\mp q)(k\pm q+1)}}_{\alpha}\langle k_1q_1,k_2q_2|kq\pm1\rangle$$

$$\langle k_1q_1,k_2q_2|\underbrace{J_{1\pm}+J_{2\pm}}_{J_\pm}|kq\rangle$$

$$\bigcirc|k,q\pm1\rangle$$

$$[J_\pm, T^k_q] = \sqrt{\alpha}\sum_{q_1,q_2}T^{k_1}_{q_1}T^{k_2}_{q_2}\langle k_1q_1,k_2q_2|k,q\pm1\rangle$$

ベクトル⊗ベクトル

$$1\otimes1 = 2\oplus1\oplus0$$

2階テンソル　　1階テンソル　　0階テンソル

　　　　　　　↳ベクトル　　　↳スカラー

$$k = k_1+k_2,\cdots,|k_1-k_2|$$

$$q: -k,\ \cdots,\ k$$

$$2k+1\text{コ}$$