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回転群の具体的な行列表示

Schwinger Boson

$$[a_\alpha, a_\beta^\dagger] = \delta_{\alpha\beta} \quad (\alpha, \beta = \pm)$$
$$[a_\alpha, a_\beta] = 0$$

$\vec{J} \equiv \frac{1}{2} a^\dagger \vec{\sigma} a$, $a = \begin{pmatrix} a_+ \\ a_- \end{pmatrix}$, $\vec{\sigma}$: パウリ行列 と定義する。

これは

$$[J_\alpha, J_\beta] = i \varepsilon_{\alpha\beta\gamma} J_\gamma$$

となるので，角運動量演算子である。

$$J_x = \frac{1}{2} (a_+^\dagger, a_-^\dagger) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_+ \\ a_- \end{pmatrix}$$

$$= \frac{1}{2} (a_+^\dagger a_- + a_-^\dagger a_+)$$

$$J_y = \frac{1}{2} (a_+^\dagger \ a_-^\dagger) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} a_+ \\ a_- \end{pmatrix}$$

$$= \frac{i}{2} (a_-^\dagger a_+ - a_+^\dagger a_-)$$

$$J_z = \frac{1}{2} (a_+^\dagger, a_-^\dagger) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a_+ \\ a_- \end{pmatrix}$$

$$= \frac{1}{2} (a_+^\dagger a_+ - a_-^\dagger a_-)$$

$$\vec{J}^2 = \hat{j}(2\hat{j} + 1)$$

$$\hat{j} = \frac{1}{2} \hat{n} = \frac{1}{2} (\hat{n}_+ + \hat{n}_-)$$

$\hat{n}_- = a_-^\dagger a_-$

$\hat{n}_+ = a_+^\dagger a_-$

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$$|n_+, n_-\rangle \equiv \frac{1}{\sqrt{n_+!}}(a_+^\dagger)^{n_+}\frac{1}{\sqrt{n_-!}}(a_-^\dagger)^{n_-}|0\rangle, \quad a_\pm|0\rangle = 0$$

$$a_+^\dagger a_+ |n_+ n_-\rangle = \frac{1}{\sqrt{n_+!}} a_+^\dagger (1 + a_+^\dagger a_+)(a_+^\dagger)^{n_+-1}\frac{1}{\sqrt{n_-!}}(a_-^\dagger)^{n_-}|0\rangle$$

$$= |n_+ n_-\rangle + \frac{1}{\sqrt{n_+!}}(a_+^\dagger)^2 a_+ (a_+^\dagger)^{n_+-1}\frac{1}{\sqrt{n_-!}}(a_-^\dagger)^{n_-}|0\rangle$$

$$= n_+ |n_+, n_-\rangle + \frac{1}{\sqrt{n_+!}}(a_+^\dagger)^{n_+} \underbrace{a_+ \frac{1}{\sqrt{n_-!}}(a_-^\dagger)^{n_-}|0\rangle}_{(a_+|0\rangle = 0)}$$

$$= n_+ |n_+ n_-\rangle$$

$$\therefore \quad \hat{n}_+ |n_+, n_-\rangle = n_+ |n_+ n_-\rangle$$

同様に $\quad \hat{n}_- |n_+, n_-\rangle = n_- |n_+, n_-\rangle$

$$J_z = \frac{1}{2}(a_+^\dagger a_- - a_-^\dagger a_-)$$

$$= \frac{1}{2}(\hat{n}_+ - \hat{n}_-)$$

$$\rightarrow \quad J_z |n_+, n_-\rangle = \frac{1}{2}(n_+ - n_-)|n_+, n_-\rangle$$

$$= m |n_+, n_-\rangle \qquad (m = \frac{1}{2}(n_+ - n_-))$$

$$j = \frac{1}{2}n = \frac{1}{2}(n_+ + n_-)$$

$$\rightarrow \quad \hat{j}(\hat{j}+1)|n_+ n_-\rangle = j(j+1)|n_+, n_-\rangle$$

ここで $\quad \begin{cases} n_+ = j + m \\ n_- = j - m \end{cases}$ である から

$$|j, m\rangle = |n_+, n_-\rangle$$

$$= \frac{(a_+^\dagger)^{j+m}(a_-^\dagger)^{j-m}}{\sqrt{(j+m)!(j-m)!}}|0\rangle$$

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$R = e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z}$ (α, β, γ : Euler角)

$R\psi = \psi D(R)$, $D(R)$: Rの表現行列

ここで

$\psi = (\psi_1, \cdots, \psi_d)$

$= (|j,-j\rangle, \cdots, |j,j\rangle)$ $d = 2j+1$

$R(|jm\rangle) = (|jm\rangle)D$ を考える.

$R|jm\rangle = |jm'\rangle D_{m'm}$

$\rightarrow D_{m'm} = \langle jm'|R|jm\rangle$ ($\langle jm'|jm\rangle = \delta_{m'm}$)

$D_{m'm} = \underbrace{\langle jm'|e^{-i\alpha J_z}}_{\fallingdotseq e^{-i\alpha m'}\langle jm'|} e^{-i\beta J_y} \underbrace{e^{-i\gamma J_z}|jm\rangle}_{\fallingdotseq |jm\rangle e^{-i\gamma m}}$

$= e^{-i\alpha m'} e^{-i\gamma m} \underbrace{\langle jm'|e^{-i\beta J_y}|jm\rangle}_{\fallingdotseq d_{m'm}(\beta)}$

$e^{-i\beta J_y}|jm\rangle = \frac{e^{-i\beta J_z}}{\sqrt{(j+m)!(j-m)!}} (a_+^\dagger)^{j+m}(a_-^\dagger)^{j-m}|0\rangle$

$= \frac{1}{\sqrt{(j+m)!(j-m)!}} e^{-i\beta J_z} a_+^\dagger \underbrace{e^{i\beta J_z} e^{-i\beta J_z}}_{=1} a_+^\dagger \cdots a_-^\dagger|0\rangle$

$= \frac{1}{\sqrt{(j+m)!(j-m)!}} [e^{-i\beta J_z} a_+^\dagger e^{i\beta J_z}]^{j+m} [e^{-i\beta J_z} a_-^\dagger e^{i\beta J_z}]^{j-m} e^{-i\beta J_z}|0\rangle$

$J_y = \frac{i}{2}(-a_+^\dagger a_- + a_-^\dagger a_+)$

↑ 右側は消滅演算子 ($a_\pm|0\rangle = 0$)

$e^{-i\beta J_y}|0\rangle = (1 + \underbrace{(-i\beta)J_y + \frac{1}{2}(-i\beta)^2 J_y^2 + \cdots}_{\rightarrow 0})|0\rangle$

$= |0\rangle$: 真空は不変

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$f_\pm(\beta) = e^{-i\beta J_y} a_\pm^\dagger e^{i\beta J_y}$ を考える。

$f_\pm(0) = a_\pm^\dagger$

$f'_\pm(\beta) = e^{-i\beta J_y}(-iJ_y) a_\pm^\dagger e^{i\beta J_y} + e^{-i\beta J_y} a_\pm^\dagger (iJ_y) e^{i\beta J_y}$

$= -i e^{i\beta J_y} [J_y, a_\pm^\dagger] e^{i\beta J_y}$

$[J_y, a_\pm^\dagger] = -\frac{i}{2}[a_+^\dagger a_- - a_-^\dagger a_+, a_\pm^\dagger]$

$= -\frac{i}{2}\{ a_+^\dagger [a_-, a_\pm^\dagger] + [a_+^\dagger, a_\pm^\dagger] a_- - a_-^\dagger [a_+, a_\pm^\dagger] - [a_-^\dagger, a_\pm^\dagger] a_+ \}$

$= -\frac{i}{2}\{ a_+^\dagger [a_-, a_\pm^\dagger] - a_-^\dagger [a_+, a_\pm^\dagger] \}$

$[J_y, a_\pm^\dagger] = \pm a_\mp^\dagger$

$\therefore f'_\pm(\beta) = \frac{1}{2} e^{-i\beta J_y}(\mp a_\mp) e^{i\beta J_y}$

$\frac{d}{d\beta}\begin{bmatrix} f_+(\beta) \\ f_-(\beta) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -f_-(\beta) \\ f_+(\beta) \end{bmatrix}$

同様に

$e^{-i\theta \hat{n}\cdot\vec{J}} \begin{pmatrix} a_+^\dagger \\ a_-^\dagger \end{pmatrix} e^{i\theta \hat{n}\cdot\vec{J}} = [E\cos\frac{\theta}{2} + i\hat{n}\cdot\vec{\sigma}\sin\frac{\theta}{2}] \begin{pmatrix} a_+^\dagger \\ a_-^\dagger \end{pmatrix}$

ここで

$\hat{n} = (0, 1, 0)$, $\theta = \beta$ とすると

$i\hat{n}\cdot\vec{\sigma} = i\sigma_y$

$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ で

$e^{-i\beta J_y} \begin{pmatrix} a_+^\dagger \\ a_-^\dagger \end{pmatrix} e^{i\beta J_y} = \begin{pmatrix} \cos\frac{\beta}{2} & \sin\frac{\beta}{2} \\ -\sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} a_+^\dagger \\ a_-^\dagger \end{pmatrix}$

$= \begin{pmatrix} a_+^\dagger \cos\frac{\beta}{2} + a_-^\dagger \sin\frac{\beta}{2} \\ -a_+^\dagger \sin\frac{\beta}{2} + a_-^\dagger \cos\frac{\beta}{2} \end{pmatrix}$

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$$e^{-i\beta J_y}|jm\rangle = \frac{1}{\sqrt{(j+m)!\,(j-m)!}}\left(a_+^\dagger\cos\frac{\beta}{2}+a_-^\dagger\sin\frac{\beta}{2}\right)^{j+m}\left(-a_+^\dagger\sin\frac{\beta}{2}+a_-^\dagger\cos\frac{\beta}{2}\right)^{j-m}|0\rangle$$

$$= \frac{1}{\sqrt{(j+m)!\,(j-m)!}}\sum_{k,l}\binom{j+m}{k}\binom{j-m}{l}(a_+^\dagger)^{j+m-k}(a_-^\dagger)^k\left(\cos\frac{\beta}{2}\right)^{j+m-k}\left(\sin\frac{\beta}{2}\right)^k$$

$$\times(-a_+^\dagger)^{j-m-l}(a_-^\dagger)^l\left(\sin\frac{\beta}{2}\right)^{j-m-l}\left(\cos\frac{\beta}{2}\right)^l|0\rangle$$

$$= \frac{1}{\sqrt{(j+m)!\,(j-m)!}}\sum_{k,l}\binom{j+m}{k}\binom{j-m}{l}(-1)^{\textcircled{1}\,j-m-l}\left(\cos\frac{\beta}{2}\right)^{\textcircled{2}\,j+m-k+l}\left(\sin\frac{\beta}{2}\right)^{\textcircled{3}\,j-m-l+k}$$

$$\times(a_+^\dagger)^{2j-k-l}(a_-^\dagger)^{k+l}|0\rangle$$

$$|jm'\rangle = \frac{1}{\sqrt{(j+m')!\,(j-m')!}}(a_+^\dagger)^{j+m'}(a_-^\dagger)^{j-m'}|0\rangle$$

内積をとると $a_+^\dagger$, $a_-^\dagger$ の数が同じところのみ残る。

$$\begin{cases} j+m' = 2j-k-l \\ j-m' = k+l \end{cases}$$

$$\longrightarrow \boxed{j-m'-k-l=0}$$

l を消す

① $j-m-l = j-m-(j-m'-k)$
$= -m+m'+k$

② $j+m-k+l = j+m-k+(j-m'-k)$
$= 2j-2k+m-m'$

③ $j-m-l+k = j-m-(j-m'-k)+k$
$= 2k-m+m'$

$\langle jm'|jm\rangle = \delta_{m'm}$ より

$\langle jm'|(a_+^\dagger)^{j+m}(a_-^\dagger)^{j-m}|0\rangle = \sqrt{(j+m')!\,(j-m')!}$

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$$d^{j}_{m'm} = \langle j\,m' | e^{-i\beta J_y} | j\,m \rangle$$

$$= \frac{1}{\sqrt{(j+m)!(j-m)!}} \sum_k (-1)^{-m+m'+k} \left(\cos\frac{\beta}{2}\right)^{2j-2k+m-m'} \left(\sin\frac{\beta}{2}\right)^{2k-m+m'} \times \sqrt{\frac{(j+m')!(j-m')!}{(j+m')!(j-m')!}} \frac{(j+m)!}{(j+m-k)!k!} \frac{(j-m)!}{(j-m-(j-m'-k))!(j-m'-k)!}$$

$$= \sum_k (-1)^{-m+m'+k} \left(\cos\frac{\beta}{2}\right)^{2j-2k+m-m'} \left(\sin\frac{\beta}{2}\right)^{2k-m+m'} \frac{\sqrt{(j+m)!(j-m)!(j+m')!(j-m')!}}{(j+m-k)!k!(k-m+m')!(j-m'-k)!}$$

和は( )!が負にならない範囲 . $j = 1/2$, 整数