量子力学3 演習6

量子ホッパ系

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1) \( \hat{J}_1 = 1, \hat{J}_2 = \frac{1}{2} \) の CG 係数

\[ m_1: 1, 0, -1, \quad m_2: \frac{1}{2}, -\frac{1}{2} \]

\[ \begin{array}{ccc}
\frac{m_1}{2} & 0 & -1 \\
\frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\
\end{array} \]

\( \hat{J}_2 = \frac{1}{2} \Rightarrow \left| \frac{3}{2}, \frac{3}{2} \right>, \left| \frac{3}{2}, \frac{1}{2} \right>, \left| \frac{3}{2}, -\frac{1}{2} \right> \)

\( \hat{J}_1 \) について \( \left| j, m \right> \)

\[ \left| j, m \right> = \left| \frac{1}{2}, \frac{1}{2} \right>, \left| \frac{1}{2}, -\frac{1}{2} \right>, \left| 0, -\frac{1}{2} \right>, \left| -1, -\frac{1}{2} \right> \]

\[ \frac{3}{2}, \frac{3}{2} \rangle = |11\rangle \otimes |\frac{1}{2}, \frac{1}{2}\rangle = |11\rangle |\frac{1}{2}, \frac{1}{2}\rangle \]

\[ \left| \frac{3}{2}, \frac{1}{2} \right> = \frac{1}{\hbar \sqrt{(\frac{3}{2} + \frac{3}{2})(\frac{3}{2} - \frac{3}{2} + 1)}} J_{-} \left| \frac{3}{2}, \frac{3}{2} \right> \]

\[ = \frac{1}{\sqrt{3} \hbar} (J_{1-} + J_{2-}) |11\rangle |\frac{1}{2}, \frac{1}{2}\rangle \]

\[ = \frac{1}{\sqrt{3} \hbar} \left\{ \hbar \sqrt{(1+1)(1-1+1)} |10\rangle |\frac{1}{2}, \frac{1}{2}\rangle + \hbar \sqrt{(\frac{3}{2} + \frac{3}{2})(\frac{3}{2} - \frac{3}{2} + 1)} |11\rangle |\frac{1}{2}, -\frac{1}{2}\rangle \right\} \]

\[ = \frac{\sqrt{2}}{3} |10\rangle \frac{1}{2}, \frac{1}{2}\rangle + \frac{1}{\sqrt{3}} |11\rangle \frac{1}{2}, -\frac{1}{2}\rangle \]
\[ \left| \frac{3}{2} - \frac{1}{2} \right> = \frac{1}{\hbar \sqrt{(\frac{3}{2} + \frac{1}{2})(\frac{3}{2} - \frac{1}{2} + 1)}} J_- \left| \frac{3}{2} \right> \]

\[ = \frac{1}{2\hbar} (J_1+J_2^-) \left( \sqrt{\frac{2}{3}} \left| 10 \frac{1}{2} \frac{1}{2} \right> + \sqrt{\frac{1}{3}} \left| 11 \frac{1}{2} - \frac{1}{2} \right> \right) \]

\[ = \frac{1}{2\hbar} \sqrt{\frac{2}{3}} (J_1-110) \left| 1 \frac{1}{2} \frac{1}{2} \right> + \frac{1}{2\hbar} \sqrt{\frac{2}{3}} 110 \left( J_2 - 1 \frac{1}{2} \frac{1}{2} \right) \]

\[ + \frac{1}{2\hbar} \frac{1}{\sqrt{3}} (J_1-111) \left| 1 \frac{1}{2} - \frac{1}{2} \right> + \frac{1}{2\hbar} \frac{1}{\sqrt{3}} 111 \left( J_2 - 1 \frac{1}{2} - \frac{1}{2} \right) \]

\[ = \frac{1}{\sqrt{6}} \sqrt{(1+0)(1-0+1)} \left| 11 \frac{1}{2} - \frac{1}{2} \right> + \frac{1}{\sqrt{6}} \sqrt{(\frac{3}{2}+\frac{1}{2})(\frac{3}{2} - \frac{1}{2} + 1)} \left| 10 \frac{1}{2} - \frac{1}{2} \right> \]

\[ + \frac{1}{\sqrt{2}} \sqrt{(1+1)(1-1+1)} \left| 10 \frac{1}{2} \frac{1}{2} \right> + \frac{1}{\sqrt{2}} \sqrt{(\frac{3}{2} - \frac{1}{2})} \]

\[ = \frac{1}{\sqrt{3}} \left| 11 \frac{1}{2} - \frac{1}{2} \right> + \frac{1}{\sqrt{6}} \left| 10 \frac{1}{2} \frac{1}{2} \right> + \frac{1}{\sqrt{6}} \left| 10 \frac{1}{2} - \frac{1}{2} \right> \]

\[ = \frac{1}{\sqrt{3}} \left| 11 \frac{1}{2} - \frac{1}{2} \right> + \frac{2}{\sqrt{6}} \left| 10 \frac{1}{2} - \frac{1}{2} \right> \]
\[
\left| \frac{3}{2} - \frac{3}{2} \right> = \frac{1}{\sqrt{\sqrt[3]{3}} \sqrt{(2x-\frac{1}{2})(\frac{3}{2}+\frac{1}{2})+1}} \left( \frac{1}{\sqrt{3}} \left| 1-\left( \frac{1}{2} \right) \right> + \frac{2}{\sqrt{6}} \left| 1-\left( \frac{1}{2} \right) \right> \right)
\]

\[
= \frac{1}{3} \left| \frac{1}{2} \right> \left( \frac{1}{2} \right) + \frac{2}{3} \left( \frac{1}{2} \right) \left| \frac{1}{2} \right> \left( \frac{1}{2} \right) + \frac{2}{3} \left| \frac{1}{2} \right> \left( \frac{1}{2} \right)
\]

次に \( \frac{1}{2} \) を考える。\( M = \frac{1}{2} \) となる \( \frac{3}{2} m1 m2 \) は

\[
\left| 0 \frac{1}{2} \right> \text{ と } \left| 1 \frac{1}{2} - \frac{1}{2} \right> \text{ であるから，}
\]

\[
\left| \frac{1}{2} \right> = a \left| 0 \frac{1}{2} \right> + b \left| 1 \frac{1}{2} - \frac{1}{2} \right> \text{ とおいて } a, b \text{ を求める。}
\]

\[
\left< \frac{1}{2} \frac{1}{2} \right> = 1 \Rightarrow a^2 + b^2 = 1
\]

\[
\left< \frac{3}{2} \frac{1}{2} \right> = 0 \Rightarrow \sqrt{\frac{2}{3}} a + \sqrt{\frac{1}{3}} b = 0
\]

\[
a^2 + 2a^2 = 1
\]

\[
a = \frac{1}{\sqrt{3}} \quad a = \pm \frac{1}{\sqrt{3}} , \quad b = \mp \sqrt{\frac{2}{3}}
\]
ここで $b > 0$ の方にすると $a = -\frac{1}{\sqrt{3}}, \ b = \frac{\sqrt{2}}{3}$

$|\frac{1}{2} \frac{1}{2}\rangle = -\frac{1}{\sqrt{3}} |10\frac{1}{2} \frac{1}{2}\rangle + \frac{\sqrt{2}}{3} |11 \frac{1}{2} - \frac{1}{2}\rangle$

$|\frac{1}{2} - \frac{1}{2}\rangle = \frac{1}{\hbar \sqrt{(\frac{1}{2} + \frac{1}{2})(\frac{1}{2} - \frac{1}{2} + 1)}} J - |\frac{1}{2} \frac{1}{2}\rangle$

$= \frac{1}{\hbar} (J_1^- + J_2^-) \left(-\frac{1}{\sqrt{3}} |10 \frac{1}{2} \frac{1}{2}\rangle + \frac{\sqrt{2}}{3} |11 \frac{1}{2} - \frac{1}{2}\rangle \right)$

$= -\frac{1}{\sqrt{3}} \sqrt{(1+0)(1-0+1)} |11 -1 \frac{1}{2} \frac{1}{2}\rangle$

$- \frac{1}{\sqrt{3}} \sqrt{(\frac{1}{2} + \frac{1}{2})(\frac{1}{2} - \frac{1}{2} + 1)} |10 \frac{1}{2} - \frac{1}{2}\rangle$

$+ \frac{\sqrt{2}}{3} \cdot \sqrt{(1+1)(1-1+1)} |10 \frac{1}{2} - \frac{1}{2}\rangle$

$= -\frac{\sqrt{2}}{3} |11 -1 \frac{1}{2} \frac{1}{2}\rangle - \frac{1}{\sqrt{3}} |10 \frac{1}{2} - \frac{1}{2}\rangle + \frac{2}{\sqrt{3}} |10 \frac{1}{2} - \frac{1}{2}\rangle$

$= -\frac{\sqrt{2}}{3} |11 -1 \frac{1}{2} \frac{1}{2}\rangle + \frac{1}{\sqrt{3}} |10 \frac{1}{2} - \frac{1}{2}\rangle$. 
以上をまとめると

\[ \frac{3}{2} \frac{3}{2} \rangle = \begin{bmatrix} 1 & 1 \end{bmatrix} \frac{1}{2} \frac{1}{2} \rangle \\
\frac{3}{2} \frac{1}{2} \rangle = \sqrt{2} \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{2} \frac{1}{2} \rangle + \sqrt{3} \begin{bmatrix} 1 \end{bmatrix} \frac{1}{2} - \frac{1}{2} \rangle \\
\frac{3}{2} - \frac{1}{2} \rangle = \sqrt{3} \begin{bmatrix} 1 & -1 \end{bmatrix} \frac{1}{2} \frac{1}{2} \rangle + \sqrt{2} \begin{bmatrix} 0 \end{bmatrix} \frac{1}{2} - \frac{1}{2} \rangle \\
\frac{3}{2} - \frac{3}{2} \rangle = \begin{bmatrix} 1 & -1 \end{bmatrix} \frac{1}{2} - \frac{1}{2} \rangle \\
\frac{1}{2} \frac{1}{2} \rangle = -\sqrt{3} \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{2} \frac{1}{2} \rangle + \sqrt{3} \begin{bmatrix} 1 \end{bmatrix} \frac{1}{2} - \frac{1}{2} \rangle \\
\frac{1}{2} - \frac{1}{2} \rangle = -\sqrt{2} \begin{bmatrix} 1 & -1 \end{bmatrix} \frac{1}{2} \frac{1}{2} \rangle + \sqrt{3} \begin{bmatrix} 0 \end{bmatrix} \frac{1}{2} - \frac{1}{2} \rangle \\
\begin{pmatrix} m_1 \; m_2 \; m_3 \end{pmatrix} の前にかけている係数が \tilde{d}_1 = 1, \, \tilde{d}_2 = \frac{1}{2} の CG 係数である。


2) (i) \begin{pmatrix} s_1 \; s_2 \; s_3 \end{pmatrix} = \begin{pmatrix} s_1 \end{pmatrix} \otimes \begin{pmatrix} s_2 \end{pmatrix} \otimes \begin{pmatrix} s_3 \end{pmatrix} \\
\begin{pmatrix} s_1 \end{pmatrix} = 1 \downarrow \cdots \downarrow 2 \text{ 通り かつ } \begin{pmatrix} s_1 \; s_2 \; s_3 \end{pmatrix} \text{ は } 2^3 = 8 \text{ 通り }
(i) \[ \vec{S}^2 = S_1^2 + S_2^2 + S_3^2 + 2(\vec{S}_1 \cdot \vec{S}_2 + \vec{S}_2 \cdot \vec{S}_3 + \vec{S}_3 \cdot \vec{S}_1) \]
\[ = \frac{3}{4} h^2 \times 3 + 2(\sim) \]
\[ H_3 = J(\sim) \]
\[ = J\left(\frac{1}{2} \vec{S}^2 - \frac{1}{2}, \frac{9}{4} Jh^2\right) \]
\[ = \frac{1}{2}(J\vec{S}^2 - \frac{9}{4} Jh^2) \]

(iii) \[ \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \]
\[ = \left\{ \left(\frac{1}{2} + \frac{1}{2}\right) \otimes \left(\frac{1}{2} - \frac{1}{2}\right) \right\} \otimes \frac{1}{2} \]
\[ = (1 \otimes 0) \otimes \frac{1}{2} \]
\[ = (1 \otimes \frac{1}{2}) \oplus (0 \otimes \frac{1}{2}) \]
\[ = \left\{ \left(1 + \frac{1}{2}\right) \oplus \left(1 - \frac{1}{2}\right) \right\} \oplus (0 + \frac{1}{2}) \]
\[ = \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{3}{2} \]
(Ⅳ) \( \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{3}{2} \)

\[ \Rightarrow \] 合成ベクトルは \( \frac{3}{2} \, m \), \( \frac{1}{2} \, m \), \( \frac{1}{2} \, m \) がある。

\[ H | \frac{3}{2} \, m \rangle = \frac{1}{2} J S^2 | \frac{3}{2} \, m \rangle - \frac{9}{8} J \hbar^2 | \frac{3}{2} \, m \rangle \]
\[ = \frac{1}{2} J \hbar^2 (\frac{3}{2})(\frac{3}{2} + 1) | \frac{3}{2} \, m \rangle - \frac{9}{8} J \hbar^2 | \frac{3}{2} \, m \rangle \]
\[ = \frac{3}{4} J \hbar^2 | \frac{3}{2} \, m \rangle = E_\frac{3}{2} | \frac{3}{2} \, m \rangle \]

\[ E_\frac{3}{2} = \frac{3}{4} J \hbar^2 \text{ (4重項) } \Rightarrow \text{ 4重縮退} \]

\[ H | \frac{1}{2} \, m \rangle = \frac{1}{2} J S^2 | \frac{1}{2} \, m \rangle - \frac{9}{8} J \hbar^2 | \frac{1}{2} \, m \rangle \]
\[ = \frac{1}{2} J \hbar^2 \frac{1}{2}(\frac{1}{2} + 1) | \frac{1}{2} \, m \rangle - \frac{9}{8} J \hbar^2 | \frac{1}{2} \, m \rangle \]
\[ = (\frac{3}{8} - \frac{9}{8}) J \hbar^2 | \frac{1}{2} \, m \rangle \]
\[ = -\frac{3}{4} J \hbar^2 | \frac{1}{2} \, m \rangle \]
\[ = E_\frac{1}{2} | \frac{1}{2} \, m \rangle \]

\[ E_\frac{1}{2} = -\frac{3}{4} J \hbar^2 \text{ (4重項) } \Rightarrow \text{ 4重縮退} \]
\begin{align*}
|\frac{3}{2} \ 0\rangle &= |\uparrow\rangle \otimes |\uparrow\rangle \otimes |\uparrow\rangle = |\uparrow\rangle |\uparrow\rangle |\uparrow\rangle \\
|\frac{3}{2} \ -\frac{1}{2}\rangle &= \frac{1}{\hbar \sqrt{(\frac{3}{2}+\frac{3}{2})(\frac{3}{2}-\frac{1}{2}+1)}} S_+ |\frac{3}{2} \ -\frac{1}{2}\rangle \\
&= \frac{1}{\sqrt{3} \hbar} (S_+ + S_2 + S_3) |\uparrow\rangle |\uparrow\rangle |\uparrow\rangle \\
&= \frac{1}{\sqrt{3}} \left( |\downarrow\rangle |\uparrow\rangle |\uparrow\rangle + |\uparrow\rangle |\downarrow\rangle |\uparrow\rangle + |\uparrow\rangle |\uparrow\rangle |\downarrow\rangle \right) \\
|\frac{3}{2} \ -\frac{1}{2}\rangle &= \frac{1}{\hbar \sqrt{(\frac{3}{2}+\frac{1}{2})(\frac{3}{2}-\frac{1}{2}+1)}} S_- |\frac{3}{2} \ -\frac{1}{2}\rangle \\
&= \frac{1}{2 \hbar} (S_- + S_2 + S_3) \frac{1}{\sqrt{3}} \left( |\downarrow\rangle |\downarrow\rangle |\uparrow\rangle + |\downarrow\rangle |\uparrow\rangle |\downarrow\rangle + |\uparrow\rangle |\downarrow\rangle |\downarrow\rangle \right) \\
&= \frac{1}{2 \sqrt{3}} \left[ |\downarrow\rangle |\downarrow\rangle |\downarrow\rangle + |\uparrow\rangle |\uparrow\rangle |\uparrow\rangle + |\downarrow\rangle |\uparrow\rangle |\downarrow\rangle + |\downarrow\rangle |\uparrow\rangle |\downarrow\rangle + |\uparrow\rangle |\downarrow\rangle |\downarrow\rangle + |\uparrow\rangle |\downarrow\rangle |\downarrow\rangle + |\downarrow\rangle |\downarrow\rangle |\downarrow\rangle + |\uparrow\rangle |\uparrow\rangle |\downarrow\rangle + |\downarrow\rangle |\uparrow\rangle |\downarrow\rangle + |\uparrow\rangle |\downarrow\rangle |\downarrow\rangle + |\downarrow\rangle |\downarrow\rangle |\downarrow\rangle \right] \\
&= \frac{1}{\sqrt{3}} \left( |\downarrow\rangle |\downarrow\rangle |\downarrow\rangle + |\downarrow\rangle |\uparrow\rangle |\downarrow\rangle + |\downarrow\rangle |\downarrow\rangle |\downarrow\rangle \right)
\end{align*}
\[ \begin{align*}
\left| \frac{3}{2} - \frac{3}{2} \right> &= \frac{1}{\sqrt{3\hbar}} \sqrt{\frac{1}{3}} \left( |\downarrow_1\rangle |\downarrow_2\rangle |\downarrow_3\rangle + |\uparrow_1\rangle |\downarrow_2\rangle |\downarrow_3\rangle + |\uparrow_1\rangle |\uparrow_2\rangle |\downarrow_3\rangle \right) \\
&= |\downarrow_1\rangle |\downarrow_2\rangle |\downarrow_3\rangle 
\end{align*} \]

次に \( \left| \frac{1}{2} \frac{1}{2} \right> \) と \( \left| \frac{1}{2} \frac{1}{2} \right> \) を考えます。

\( \left| \frac{3}{2} \frac{1}{2} \right>, \left| \frac{1}{2} \frac{1}{2} \right>, \left| \frac{1}{2} \frac{1}{2} \right> \) は基底 \( \left| \downarrow_1\rangle \right| \left| \downarrow_2\rangle \right| \left| \downarrow_3\rangle \right>, \left| \uparrow_1\rangle \right| \left| \downarrow_2\rangle \right| \left| \downarrow_3\rangle \right>, \left| \downarrow_1\rangle \right| \left| \uparrow_2\rangle \right| \left| \downarrow_3\rangle \right> \) の線形結合で表されます。

\( \left| \frac{1}{2} \frac{1}{2} \right> \) は \( |\downarrow \rangle = \alpha \left| \frac{1}{2} \right>, \left| \uparrow \rangle = \beta \left| \frac{1}{2} \right> \) と \( \left| \downarrow \rangle \left| \uparrow \rangle \right| \left| \downarrow \rangle \right> \) の線形結合で表されます。

\( \left| \frac{3}{2} \frac{1}{2} \right> = \frac{1}{\sqrt{3}} (\alpha + \beta + \gamma) \)

\( \left| \frac{1}{2} \frac{1}{2} \right> = A (-\alpha - \beta + 2\gamma) \)

\( A: \) 定数

\( \left< \frac{1}{2} \frac{1}{2} \left| \frac{1}{2} \frac{1}{2} \right> \right> = 1 \) なので \( A = \frac{1}{\sqrt{6}} \)

\( \therefore \left| \frac{1}{2} \frac{1}{2} \right> = \frac{1}{\sqrt{6}} (-\alpha - \beta + 2\gamma) \)
\[ |1/2 \ 1/2\rangle_2 = (d\alpha + e\beta + f\gamma) \text{ とおく} \]

\[ d, e, f \text{ は定数} \]

\[ \langle 1/2 \ 1/2 | 1/2 \ 1/2 \rangle_2 = d^2 + e^2 + f^2 = 1 \]

\[ \langle 3/2 \ 1/2 | 1/2 \ 1/2 \rangle_2 = d + e + f = 0 \]

\[ \Rightarrow f = 0, \quad d = \frac{1}{\sqrt{2}}, \quad e = -\frac{1}{\sqrt{2}} \]

\[ \langle 1/2 \ 1/2 | 1/2 \ 1/2 \rangle_1 = \frac{1}{\sqrt{6}} \left( -|\downarrow_1\rangle |\uparrow_2\rangle |\uparrow_3\rangle - |\uparrow_1\rangle |\downarrow_2\rangle |\uparrow_3\rangle + 2 |\uparrow_1\rangle |\uparrow_2\rangle |\downarrow_3\rangle \right) \]

\[ |1/2 \ 1/2\rangle_2 = \frac{1}{\sqrt{2}} \left( |\downarrow_1\rangle |\uparrow_2\rangle |\uparrow_3\rangle - |\uparrow_1\rangle |\downarrow_2\rangle |\uparrow_3\rangle \right) \]

\[ |1/2 - 1/2\rangle_1 = \frac{1}{\sqrt{6}} \left( -|\downarrow_1\rangle |\downarrow_2\rangle |\uparrow_3\rangle + 2 |\downarrow_1\rangle |\uparrow_2\rangle |\downarrow_3\rangle \right) \]

\[ = \frac{1}{\sqrt{6}} \left( |\downarrow_1\rangle |\uparrow_2\rangle |\uparrow_3\rangle + |\uparrow_1\rangle |\downarrow_2\rangle |\downarrow_3\rangle - 2 |\downarrow_1\rangle |\uparrow_2\rangle |\uparrow_3\rangle \right) \]

\[ |1/2 - 1/2\rangle_2 = \frac{1}{\sqrt{6}} \left( -|\downarrow_1\rangle |\downarrow_2\rangle |\uparrow_3\rangle + |\downarrow_1\rangle |\uparrow_2\rangle |\uparrow_3\rangle + |\downarrow_1\rangle |\uparrow_2\rangle |\downarrow_3\rangle \right) \]

\[ = \frac{1}{\sqrt{2}} \left( -|\downarrow_1\rangle |\downarrow_2\rangle |\uparrow_3\rangle + |\downarrow_1\rangle |\uparrow_2\rangle |\uparrow_3\rangle - |\uparrow_1\rangle |\downarrow_2\rangle |\downarrow_3\rangle \right) \]
\[
\begin{align*}
\frac{1}{2} - \frac{1}{2} \parallel_2 &= \frac{1}{\sqrt{2}} (|\uparrow\rangle |\uparrow_2\rangle |\uparrow_3\rangle - |\uparrow\rangle |\downarrow_2\rangle |\downarrow_3\rangle) \\
\text{以上から } H\text{ の固有関数は} \quad &\begin{align*}
|\frac{3}{2}, \frac{3}{2}\rangle &= |\uparrow\rangle |\uparrow_2\rangle |\uparrow_3\rangle \\
|\frac{3}{2}, \frac{1}{2}\rangle &= \frac{1}{\sqrt{3}} (|\uparrow\rangle |\uparrow_2\rangle |\uparrow_3\rangle + |\uparrow\rangle |\downarrow_2\rangle |\uparrow_3\rangle + |\uparrow\rangle |\downarrow_2\rangle |\downarrow_3\rangle) \\
|\frac{3}{2}, -\frac{1}{2}\rangle &= \frac{1}{\sqrt{3}} (|\uparrow\rangle |\downarrow_2\rangle |\downarrow_3\rangle + |\uparrow\rangle |\downarrow_2\rangle |\uparrow_3\rangle + |\downarrow\rangle |\downarrow_2\rangle |\uparrow_3\rangle) \\
|\frac{3}{2}, -\frac{3}{2}\rangle &= |\downarrow\rangle |\uparrow_2\rangle |\downarrow_3\rangle \\
|\frac{1}{2}, \frac{1}{2}\rangle_1 &= \frac{1}{\sqrt{6}} (-|\downarrow\rangle |\uparrow_2\rangle |\uparrow_3\rangle - |\uparrow\rangle |\downarrow_2\rangle |\uparrow_3\rangle + 2 |\uparrow\rangle |\uparrow_2\rangle |\downarrow_3\rangle) \\
|\frac{1}{2}, -\frac{1}{2}\rangle_1 &= \frac{1}{\sqrt{6}} (|\downarrow\rangle |\uparrow_2\rangle |\uparrow_3\rangle + |\uparrow\rangle |\downarrow_2\rangle |\downarrow_3\rangle - 2 |\uparrow\rangle |\downarrow_2\rangle |\uparrow_3\rangle) \\
|\frac{1}{2}, \frac{1}{2}\rangle_2 &= \frac{1}{\sqrt{2}} (|\uparrow\rangle |\uparrow_2\rangle |\uparrow_3\rangle - |\uparrow\rangle |\downarrow_2\rangle |\uparrow_3\rangle) \\
|\frac{1}{2}, -\frac{1}{2}\rangle_2 &= \frac{1}{\sqrt{2}} (|\downarrow\rangle |\uparrow_2\rangle |\uparrow_3\rangle - |\uparrow\rangle |\downarrow_2\rangle |\downarrow_3\rangle)
\end{align*}
\end{align*}
\]
3

(1) \[ \vec{S} = \vec{S}_1 + \vec{S}_2 \]
\[ \vec{S}^2 = S_1^2 + S_2^2 + 2 \vec{S}_1 \cdot \vec{S}_2 \]
\[ = 2 \hbar^2 S (S+1) + 2 \cdot \frac{H_S}{J_1} \]
\[ H_S = \frac{1}{2} J_1 \vec{S}^2 - \hbar^2 J_1 S (S+1) \]

(11) \[ S \otimes S = 2 S \otimes 2 S - 1 \otimes \ldots \otimes 0 \]
\[ \vec{S}^2 = \hbar^2 S' (S'+1) \]
\[ H_S |S_1 S_2\rangle = \left[ \frac{1}{2} J_1 \hbar^2 S' (S'+1) - \hbar^2 J_1 S (S+1) \right] |S_1 S_2\rangle \]
\[ = E^S |S_1 S_2\rangle \]

エネルギー
\[ E^S = \frac{1}{2} J_1 \hbar^2 S' (S'+1) - \hbar^2 J_1 S (S+1) \]

縮退度は \[ 2S'+1 \quad (S' = 2S, 2S-1, \ldots, 0) \]
(iii) \[ H_m^S = J_m (\vec{S}_1 \cdot \vec{S}_2)^m \]
\[ \vec{S}^2 = S_1^2 + S_2^2 + 2 S_1 S_2 \]
\[ \left( \frac{1}{2} \vec{S}^2 - \hbar^2 S (S+1) \right)^m = (\vec{S}_1 \cdot \vec{S}_2)^m = \frac{H_m^S}{J_m} \]
\[ \Rightarrow E_m^S = J_m \left( \frac{1}{2} \hbar^2 S' (S' + 1) - \hbar^2 S (S+1) \right)^m \]

縮退度は \( 2 S' + 1 \) （\( S' = 2S, 2S-1, \ldots, 0 \)）

(4)

(i) \[ H_N = J \sum_{i,j=1}^{N} \vec{S}_i \cdot \vec{S}_j \quad \vec{S}^2 = \frac{3}{4} \hbar^2 \]
\[ \vec{S}^2 = \sum_{i=1}^{N} \vec{S}_i^2 + 2 \sum_{i \neq j} \vec{S}_i \cdot \vec{S}_j \]
\[ = \frac{3}{4} N \hbar^2 + 2 \cdot \frac{H_N}{J} \]
\[ \Rightarrow H_N = \frac{J}{2} \left( \vec{S}^2 - \frac{3}{4} N \hbar^2 \right) \]
(i) \( S_r = \pm \frac{1}{2} \rightarrow 2 \)通り
\[
2 \times 2 \times \cdots \times 2 = 2^N \text{個}
\]

(\text{Nコ})

(iii) \( \text{Nコのうちmコを下に変えたものであるので} \)
\[
S_k | S_1, S_2, \ldots, S_N > = \frac{1}{\lambda} (\frac{N-m}{2} - \frac{m}{2}) | S_1, \ldots, S_N >
\]
\[
= \frac{1}{\lambda} (\frac{N}{2} - m) | S_1, \ldots, S_N >
\]

\[
\text{DM} = \text{NCm} = \frac{N!}{(N-m)!m!}
\]
\[
= \frac{N!}{\{N-(\frac{N}{2}-M)^2\}!(\frac{N}{2}-M)!}
\]
\[
= \frac{N!}{(\frac{N}{2}+M)!(\frac{N}{2}-M)!}
\]
電子数と合成スピンの関係は

\[ \begin{align*}
S & \rightarrow \text{粒子数} \ N \\
1 & \rightarrow 0 \\
\frac{3}{2} & \rightarrow 1 \\
\frac{1}{2} & \rightarrow 2 \\
0 & \rightarrow 3 \\
\end{align*} \]

\[ d_{N/2} = D_{N/2} = \frac{N!}{N!} = 1 \]

\[ d_{N/2-m} (1 \leq m \leq N/2) = D_{N/2-m} = \frac{N!}{(N/2+m)! \cdot (N/2-m)!} - \frac{N!}{(N/2+m+1)! \cdot (N/2-m-1)!} \]

\[ = \frac{N!}{(N/2+S)!(N/2-S)!} - \frac{N!}{(N/2+S+1)!(N/2-S-1)!} \]

となっている。
(V) \[ \sum_{s=0}^{N} (2s+1) d_s = (N+1) D_2 + \sum_{s=0}^{N-1} (D_s - D_{s+1})(2s+1) \]

\[ = N \frac{D_{N/2} + D_{N/2} - D_0 + D_1 + 3D_1 - 3D_2 + 5D_2 - 5D_3 + \cdots}{2} + N \frac{D_{N/2} - 2 - ND_{N/2} - 1 - 3D_{N/2} - 2}{2} \]

\[ + 3D_{N/2} - 1 + ND_{N/2} - D_{N/2} - 1 - ND_{N/2} \]

\[ + D_{N/2} \]

= \[ 2 \sum_{s=1}^{N/2} D_s + D_0 \]

\[ \downarrow \]

これは全状態の数を表すので、\( 2^N \)と等しい。

\[ \sum_{s=0}^{N/2} (2s+1) d_s = 2^N \]

\[ \downarrow \]

\( d_s \)は\( |5s> \)の数を表すので、それらを2s+1をかけて\( |5s> \)の全状態数を表す。

\[ \sum_{s=0}^{N/2} (2s+1) d_s = |5s\ldots 5n> \]

の全状態数を表す。
$H_N | S_i \cdots S_n \rangle = \frac{J}{2} S^2 | S_i \cdots S_n \rangle - \frac{3}{8} J N \hbar^2 | S_i \cdots S_n \rangle$

$$= \left[ \frac{J}{2} \hbar^2 S(S+1) - \frac{3}{8} J N \hbar^2 \right] | S_i \cdots S_n \rangle$$

$E_N = \frac{J}{2} \hbar^2 S(S+1) - \frac{3}{8} J N \hbar^2$

縮退度 $(2S+1) \, ds \quad (S = 0, \cdots, \frac{N}{2})$